PHILOSOPHY OF MATHEMATICS EDUCATION

by

E R I C B L A I R E

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ABSTRACT

PHILOSOPHY OF MATHEMATICS EDUCATION

This thesis supports the view that mathematics teachers should be aware of differing views of the nature of mathematics and of a range of teaching perspectives. The first part of the thesis discusses differing ways in which the subject 'mathematics' can be identified, by relying on existing philosophy of mathematics. The thesis describes three traditionally recognised philosophies of mathematics: logicism, formalism and intuitionism. A fourth philosophy is constructed, the hypothetical, bringing together the ideas of Peirce and of Lakatos, in particular.

The second part of the thesis introduces differing ways of teaching mathematics, and identifies the logical and sometimes contingent connections that exist between the philosophies of mathematics discussed in part 1, and the philosophies of mathematics teaching that arise in part 2. Four teaching perspectives are outlined: the teaching of mathematics as aesthetically-orientated, the teaching of mathematics as a game, the teaching of mathematics as a member of the natural sciences, and the teaching of mathematics as technology-orientated. It is argued that a possible fifth perspective, the teaching of mathematics as a language, is not a distinctive approach. A further approach, the Inter-disciplinary perspective, is recognised as a valid alternative within previously identified philosophical constraints.

Thus parts 1 and 2 clarify the range of interpretations found in both the philosophy of mathematics and of mathematics teaching and show that they present realistic choices for the mathematics teacher. The foundations are thereby laid for the arguments generated in part 3, that any mathematics teacher ought to appreciate the full range of teaching

perspectives which may be chosen and how these link to views of the nature of mathematics. This would hopefully reverse 'the trend at the moment... towards excessively narrow interpretation of the subject' as reported by Her Majesty's Inspectorate (Aspects of Secondary Education in England, 7.6.20, H.M.S.O., 1979).

While the thesis does not contain infallible prescriptions it is concluded that the technology-orientated perspective supported by the hypothetical philosophy of mathematics facilitates the aims of those educators who show concern for the recognition of mathematics in the curriculum, both for its intrinsic and extrinsic value. But the main thrust of the thesis is that the training of future mathematics educators must include opportunities for gaining awareness of the diversity of teaching perspectives and the influence on them of philosophies of mathematics.

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CONTENTS

INTRODUCTION

During the last twenty-five years there have been radical changes in the mathematics taught in primary and secondary schools, and in universities and other institutions of higher education. If one had been studying for a first degree in mathematics just ten years ago, one's first acquaintance with the term 'set', would probably have come in the first or even the second year of that degree. To-day, a five or six year-old may be introduced to terms like 'set' and 'intersection' in his first year of primary education. This is indicative of the kind of change that has occurred in the last two decades. It is likely that twenty-five years ago the only people acquainted with the language of set theory as students, were those that chose such or related areas of specialisation at higher degree level. Mathematics education has been a growing focus for debate in the last decade particularly, but while practising teachers and several schools of psychology, like those following Piaget, Dienes and more recently Skemp, have written extensively on the subject of mathematics education, philosophers have generally chosen to remain silent on the subject.

Philosophers may have believed that they have nothing useful to say on the subject of mathematics education. If this is the case, then this thesis attempts to prove them wrong. The intention is to break the silence and to lay out for mathematics educators, at whatever level they may teach, the diverse nature of both mathematics and mathematics teaching, as philosophical analysis identifies it. By clarifying existing distinctions and by introducing some new ones, the intention of this thesis is to show that philosophy can usefully contribute to the ongoing debate on mathematics education. Not all the distinctions are new, but by showing their relevance to mathematics education, it is hoped that many people involved in mathematics who were previously oblivious to the distinctions will come to recognise their relevance to them. Some of these distinctions have previously been hidden in philosophy of mathematics courses, and others have only arisen since many practising mathematics teachers were trained. Furthermore, even those that have experienced much of the contemporary debate will have also felt the influence of various writers on the curriculum, most notably of whom has been Professor Hirst. Hirst argues that there are 'distinct disciplines or forms of knowledge' and in particular, that 'mathematics depends on deductive demonstrations from certain sets of axioms' ('Liberal Education and the Nature of Knowledge' in Philosophical Analysis and Education, pages 113 to 138), and that this tidily packages away mathematics so that one can go on to discuss other disciplines, with greater ambiguity. That the packaging of mathematics is not so tidy was a major stimulus to the writing of this thesis, and it is to be hoped that the thesis at least succeeds in showing that it is worthwhile looking in depth at just one of those 'forms of knowledge', through a philosophical microscope. It is not that the thesis attempts to challenge the work of Hirst or any other writer in this area, but simply tries to demonstrate the value of putting just one discipline under the microscope. By narrowing down the area of attention in this way, it has been possible to look at not only the nature of mathematics, but the teaching of mathematics, and the training of teachers of mathematics.

The thesis is divided into three parts corresponding to the three factors just mentioned. The first part considers the nature of mathematics; the second identifies differing approaches to mathematics teaching; and the third part moves from a consideration of the aims of mathematics education to suggesting prerequisites for being a mathematics educator, in so far as philosophical analysis can identify such features. Because of personal experience, the examples used in the thesis are generally from secondary schools, but much of what is written here could apply to teaching either younger or older students.

The overall coherence of the thesis is demonstrated by the continual interplay that is found between views given in one part of the thesis and arguments developed in another. While part 2 of the thesis is predominantly concerned with identifying differing approaches to mathematics teaching, logical and sometimes contingent links are shown to exist between particular approaches and the philosophies of mathematics, introduced in part 1. Inevitably, the discussion of mathematics teaching in part 2, leads to comments being made on differing notions of mathematics education that may seem to underpin particular approaches. In this way, part 2 may give a prior glimpse of many of the problems more fully developed in the final part of the thesis. It is also the intention of this final part, part 3, to show in particular that a philosophical critique like this can identify pointers that can be realistically considered by practising teachers, and teachers of teachers. Overall it is hoped that anyone reading this thesis will recognise the importance to mathematics education, of:

a) different views about the nature of mathematics;

b) genuine alternatives in the approaches to teaching mathematics;

c) interconnections between the nature of mathematics and the ways it is taught; and

d) the pre-requisites of teaching mathematics that philosophical analysis can identify, given that there are limits to what philosophy can appropriately comment on.

The thesis begins with an historical review of those authors that seem most clearly to foreshadow the philosophical movements in mathematics, that have developed in the last hundred years. These philosophical predecessors stretch back to Plato and forward to the work of Kant. As with any selection from more than 2000 years of writings, one could be criticised for leaving out or preferring one philosophy to another. It is hoped that the coherence of the thesis will demonstrate that the initial choice was correct, although other thinkers, like Mill and Locke are mentioned in the main body of the thesis, as they are seen to reflect or develop the views of the four philosophers given special attention in chapter 1 of the thesis, Plato, Aristotle, Leibniz and Kant. The argument in favour of this selection and also of the selection of the philosophies of mathematics discussed in the proceeding chapters, has been the influence that each can be shown to have had, not always directly, on mathematics education during the last twenty-five years. Other schools of thought and even other branches within the movements selected can be identified, but they have been excluded if there did not seem to be clear evidence that they were presently influencing mathematics education. While this is a contingent matter, it is to be hoped that the rightness of the selection does not play a critical part in the validity of the overall argument of the thesis.

While each movement is discussed separately, it is argued that due to its complexity and significance across the philosophies of mathematics, mathematical truth is not treated within each philosophy but as a separate topic in a final chapter to part 1. However, the implications of points made in that chapter, stretch throughout the thesis, and are not found just in the first part. Although part 1 includes the argument that there are various views of the nature of mathematics, it could not be claimed that what is provided there is a comprehensive introduction to the philosophy of mathematics. Parts 1 and 2 are completed by Conclusions which attempt to set the scene for the next part, and to review briefly the argument of that part. It is to be hoped that these few comments have set the scene not only for part 1 of this thesis, but also for the thesis as a whole.

PART 1: PHILOSOPHIES OF MATHEMATICS

C H A P T E R 1

WHAT IS MATHEMATICS?

INTRODUCTION

While the concentration of this first part of the thesis will be on the philosophies of mathematics developed since Frege, these philosophies were influenced by earlier presentations of related and sometimes identical questions. Two such questions are taken as central to the discussions that occur in this and the next three chapters. These two questions are:

1) Are there eternal objects of mathematics?

2) Is 'mathematics' logically distinct from the 'empirical sciences'?

The responses to these two questions as they occur in the works of Plato, Aristotle, Leibniz and Kant[1] will provide the body of this chapter, but connections will be noted between these responses and those of contemporary philosophers of mathematics. By putting the focus on a third question, the ideas of the first four chapters are brought together in the final chapter of this part of the thesis. This third question is:

3) What is identified by the phrase 'mathematical truth' and how is 'mathematical truth' demonstrated?

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1. Four philosophers are chosen to represent ideas produced over 2000 years. Plato and Aristotle are chosen because they set up the fundamental alternatives to which all later philosophers of mathematics have returned. Leibniz is chosen because he influences the foundation of contemporary positions and incorporates views on both Plato and Aristotle and also on significant contemporaries like Descartes, Hobbes and Newton. Kant is similar to Leibniz as he lays future foundations and enounces the views of contemporaries, particularly Hume. Other philosophers are noted where their influence on later thinkers is explicit (see, mention of Locke's influence on Peirce in Chapter 4, and of Mill's attempt to reduce Mathematics to an empirical science and how it contrasts with the views of Popper and Lakatos).

Through the responses to these three questions, a set of possible models of mathematics will be built up and it will be the objective of the second part of the thesis to see how each model influences the teaching of mathematics. For example, if one argues that mathematics is highly abstract and to be mathematically educated one must have first-hand appreciation of its abstract nature then this will have implications for the teaching of mathematics. There is an obvious correspondence between these three questions and the criteria given by Paul Hirst of a 'Form of Knowledge'.[1] Hirst is concerned to identify 'categorial concepts', 'unity formed by "relationships with particular social patterns"', and 'truth criteria'.[2] Like the questions given above, the result is to outline:

1) The ontological status of ultimate concepts - in this case, the foundations of mathematics.

2) The discreteness or otherwise of the discipline which may or may not lead one to give it major importance in establishing a curriculum - in this case, reasons will be developed for the position of mathematics in the curriculum.

3) The epistemological nature of the discipline - in this case, how one can demonstrate the truth of a mathematical proposition.

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1. See P. H. Hirst, 'Liberal Education and the Nature of Knowledge', in Philosophical Analysis and Education, edited by R. D. Archambault. On the question of discreteness, it is not assumed that 'logic and mathematics' are to be taken as one, with regard to question 2). The relationship between 'mathematics' and 'logic' is discussed where appropriate.

2. In a letter, published in the Journal of Further and Higher Education, 4(1), Spring 1980, Hirst spells out the main points of his theory of knowledge and that it rests in 'humanly created social "forms of life"' and not a transcendental reality. Furthermore, 'It is not the search for complex family resemblances that helps to demarcate the forms of knowledge. It is by the drawing of lines in terms of mutually irreducible truth criteria that we can make progress, I think. It is primarily by looking at differences that we can distinguish the different families, not by looking for subtle and often elusive resemblances.' (pp. 122-3).

The separation of the epistemological question for treatment in a separate chapter, and related appendix, is justified by the complexities of responses, very few of which can be understood in isolation from earlier developments, and the inappropriateness of identifying 'camps' on this issue. For example, Quine and Lakatos both reject 'analytic truth', but Lakatos is as strongly against Quine's logic-based version of Pragmatism as he is, against any metamathematics. Furthermore, truth has been a central issue again, in the last two or three decades, rather than at the time of those who founded the movements to be discussed in Chapters 2, 3 and 4. This is not to exclude some discussion of epistemological questions in these chapters.[1] This separation may be justified from another direction. There is a need to present a set of clear responses to the ontological question first, because several philosophers rest their explanations of the epistemological problem, logically, on their ontological positions. Plato's 'realism' supports his 'correspondence theory of truth' and two thousand years later, Hilbert's 'realism' supports his 'coherent theory of proof'.

Before one attempts to discuss possible responses to the three questions identified on the previous page, it is important to clarify the ways in which certain terms will be used in this thesis, in order to reduce ambiguity.

Realism and Idealism: The words 'realism' and 'idealism' may be preceded either by the adjective 'ontological' or 'epistemological'. Furthermore, someone may hold differing views according to the context being considered, and so it should be assumed throughout this thesis, unless explicit mention

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1. Dummett argues in his article, 'The Philosophical Basis of Intuitionistic Logic' that an answer to the ontological question is necessarily incomplete, prior to considering an answer to the epistemological question (Truth and other Enigmas, p. 230). Thus the conclusion to Chapter 5 has the function of drawing together the arguments of earlier chapters, as well as the epistemological points made in Chapter 5 itself.

is made to the contrary, that the context considered is the mathematical one. Thus,

Ontological Realism is to be taken as any theory that asserts the independent existence of objects (i.e. in no sense man-made), which man can experience directly, either through his senses or through his reason.

Epistemological Realism is to be taken as any theory that asserts some truths hold independently of any particular mind.

Ontological Idealism and epistemological Idealism are to be taken as theories where dependence, rather than independence is asserted, in each of the previous definitions.

QUESTION ONE: Are there eternal objects of mathematics?

Plato has one reply for most of his life, and a modified position towards the end of his life. As Plato explains in the Republic and in the Phaedo, experience of reality occurs when the mind apprehends the Forms. Mathematical objects are Forms. That is to say that there is a form 'one-ness', 'two-ness', and so on. These entities are precise, timeless and independent but limited in number. It is this last consideration that leads Plato to indicate the possibility of changes in his theory, in the Timaeus, the Laws[1] and the Philebus. In a simple mathematical proposition like '1 + 1 = 2', both 1's refer to the same 'oneness' and so it is difficult to conceive of the independent existence of the reality to which the mathematical proposition corresponds. In other words, the Platonic heaven has only one '1'. The solution chosen by Plato is to posit an intermediary world between 'sensible world' and the 'reality of the Forms' where there are mathematical objects in abundance. Multitudes of 1's and so on, exist as 'intermediate mathematical objects'.[2]

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1. Compare Plato's view of Mathematics in Phaedo 101A-102B with his later view in Philebus 16C-18D. See discussion of this change in Gulley, pp. 172-86.

2. This phrase is referred to by Gulley as used by Aristotle in the Metaphysics to explain Plato's change of view.

Leibniz discusses this same problem, but provides his own response.

> When two and two are said to make four, the latter two must be different from the former. If they were the same, nothing new would result; it would be just as if, for a joke, I wanted to make six eggs out of three by first counting three eggs, then taking away one and counting the remaining two, and finally taking one away again and counting the remaining one. But in the calculus of numbers and magnitudes, A, B or other signs do not stand for a certain thing, but for any thing of the same number of congruent parts. For any two feet are signified by 2, if a foot is the unit or measure, whence 2 + 2 makes something new, 4, and 3 by 3 makes something new, 9; for it is presupposed that what are used are always different (though of the same magnitude).
> (Leibniz : Logical Papers, p. 143 (from Gerhardt, vii, 246).

This seems to me to encapsulate Leibniz's rejection of 'Platonic Forms' and its corresponding 'realm of mathematical objects'. Truths of reason which include those of mathematics are 'intensional propositions' rather than 'extensional propositions', truths of fact. 'Extensional propositions' employ concepts that are defined by man, but do not necessarily identify an object, and so one must look to the external world for an answer.[1]

In this extract from Leibniz's works one has the critical clues to his Ontological Idealism. '2 + 2 = 4' means:

1) The concept of 4 INCLUDES the concept of 2 + 2.

2) Reference is to concepts and not to objects: these concepts are mental constructs: so, '2 + 2 makes something new, 4'.

Here there is the possibility of complete analysis (1) and complete synthesis (2). In the case of 'two feet of material bought to-day and

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1. Leibniz was particularly concerned to distinguish 'extension' from 'intension'. In his view, mathematical concepts are intensional, and like any 'truth of reason' can be completely defined by man. That is, given any 'new property', a man can know whether or not it is included within that complete definition of the concept, e.g. Is '6' a factor of '36'?: the answer is necessarily, 'yes', for 36 = 6 x 6, 4 x 9, and so on, as defined within arithmetic. On the other hand, if one is asked 'Does Joe eat cabbages?', then there is no way one can automatically search through the definition of 'Joe' to find the property, 'eats cabbages', for it is a matter of fact. In Physics one may posit 'pulsa' and employ the concept in propositions. The truth of the propositions remains empirical.

two feet bought later', the total making 'four feet' is not a truth of reason, for no man has the complete conception of material, that would be the case in 'intensional' examples. Here one must refer to a world independent of man and not just to 'ideas'. There is a modification of 'truths of fact' in Leibniz's later work, where he suggests that they be treated as 'hypothetical propositions' (Leibniz: Logical Papers, p. 121) on the basis that if certain conditions are known to God and assumed by man, then the truth follows necessarily. Thus, if material does subdivide or join together, 'two feet of material bought to-day and two feet bought later will make four feet'. Leibniz rejects the Platonic 'reality' but retains the 'analytic a priority' of mathematical propositions.[1] Leibniz was aware of certain difficulties in the reference of true propositions well before Frege was to stumble because of them.[2] Leibniz uses the example, 'Every perfectly good man is happy' to indicate an analytic proposition whose truth is independent of the existence of the subject, 'good man'. It is this 'ideal' feature that makes the proposition not just analytic, but analytic a priori.[3] Similarly, 1,000,000 + 2,000,000 = 3,000,000 is true whether or not 3,000,000 of things exist anywhere.

If one were looking at Leibniz's total ontology then further parallels with Plato could be drawn, but Leibniz's monads identify individual substances, while Forms are universals. However, their agreement on the existence of a transcendental reality is not to contradict their disagreement as to whether or not mathematical objects belong to that reality. A similar partial agreement and disagreement occurs between Aristotle and Plato.

Aristotle discusses, much more fully than Plato had done himself,

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1. The implications for mathematical truth of these comments will be fully discussed in Chapter 5.
2. See Chapter 2 for details of Frege's position and criticisms of it. Some indications were given on p.15 above.
3. 'Analytic a priori' may be taken as synonymous with 'truth of reason'.

Plato's later modifications of his view of mathematics. Aristotle sees these modifications as a partial admission that the earlier view of the Forms[1] was inadequate. For Aristotle, the example, '1,000,000 + 2,000,000 = 3,000,000' is a proposition which belongs to 'Universal Mathematics' whose ontological status is ideal. '1,000,000' has no ontological reality apart from '1,000,000 Welshmen' say, but one may pretend such a separation can be made. One posits an independent existence for the number. However this idealism is not to be confused with the epistemological realism inherent in the question, 'what is 1,000,000 + 2,000,000?'. The answer '3,000,000' is universally and eternally true, but there is no need to match eternal forms to demonstrate this. What are comparable to Platonic Forms in Aristotle's theory are his 'categories'. Among these categories is that of 'measure' or 'quantity'. This category is exemplified when one asks, 'what is the measure of that cat?', and one replies, 'it is two feet long'. The point about a category is that if it makes sense to ask of some term in ordinary language, 'what is that?', then it is always possible to answer, by making linkings with each of the categories. Thus the cat is not just measurable, but has 'quality', in being a 'black' cat. While Aristotle's theory of categories involves assertion of epistemological realism, it does not entail any ontological commitments. Thus the categorial analysis is as appropriate for a 'Cheshire cat' as 'the cat at No. 3', without one believing that 'Cheshire cat' has more than fictional reality.[2]

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1. Reference is again being made to reports of Plato's later work as represented by Aristotle in his <u>Metaphysics</u>, and discussed by writers on Plato, like Gulley.

2. What arises is the choice between empiricist interpretations of Aristotle and more Platonic ones. As A. E. Taylor says (<u>Aristotle</u>, p. 54), 'he seems to be an empiricist or a Platonist according as you choose to remember one-half of his statement or the other.' (The Empiricist interpretations of Aristotle echo strongly Locke's distinction between 'idea' and 'quality', which leads Locke to the idealism, one finds in <u>Essay Concerning Human Understanding</u>, Book II, Chapter 8, 'ideas are the resemblances of something really existing in the objects themselves. But...the quality produced hath commonly no resemblance with anything in the thing producing it...' (pp. 72-3)). Aristotle may be seen as placing each concept under various categories, in order to define the term, i.e. answer the question, 'what is that?'.

While some interpret Aristotle's theory of universal mathematics as based upon 'abstraction from sense experience', others like Koerner, prefer, 'A variant of this interpretation (which) would be to say that the empirical apple is one, in the sense that it is a member of the class of mathematical units, just as it is red in the sense that it is a member of the class of red things.' (p. 19).

On all interpretations it is agreed that Aristotle rejects the independent existence of 'forms' and explains the non-existence of mathematical objects in terms of 'abstraction' or 'idealization'.[1] Aristotle describes a system of mathematics as that created and apprehended by the reasoning of mathematicians according to the rules of logic.[2] Like Plato's and Leibniz's, this system would consist of analytic a priori propositions. Aristotle may be seen therefore as merely modifying Plato's ontological position along lines that Aristotle seems to suggest Plato had himself tentatively considered.

Kant has a partial agreement with Plato. However, Kant distinguishes 'space and time' as parts of the categorial framework by which all

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1. Koerner prefers 'idealizing abstraction' but Miss Anscombe seems to suggest that what Aristotle indicates are 'definitions'. Thus, what is common to all examples of '24' is the definition by which '24' is identified, and the definition has no 'per se existence' but in its successful identification 'in substance'. Given Miss Anscombe's unhappiness at Russell not recognising similarities between his work in the area of 'definite descriptions' and Aristotle's Categories she may be suggesting that Aristotle is leaning towards 'Nominalism', as Plato set the way to 'Realism'. For 'Nominalism' the word 'unicorn' is a sign of something meaningful in a mind but in no way indicates an existent 'reference' (Miss Anscombe's contribution on 'Aristotle' in 3 Philosophers).

2. In answering the second question it will be seen that Aristotle makes it clear that all science, including mathematics is a deductive system, but only mathematics and its propositions are produced by 'idealizing abstraction' and established by 'intellectual intuition' (O'Connor's phrase) independently of qualities of substances; i.e. 'thought' alone (see pp.21-2 below). Signs of Aristotle laying the path to Kant's synthetic a priori notion are indicated particularly in the De anima, where for example, one finds, 'when we think of mathematical objects we conceive them, though not in fact separate from matter, as though they were separate'.

knowledge and understanding is made possible, from mathematics as 'pure natural science'. This distinction resembles Aristotle's distinction between 'measure' as a category and 'universal mathematics'. As was mentioned above, it is left to Aristotle to make claims that Plato had also recognised this type of distinction. Certainly, mathematics has a central role in Plato's epistemology, throughout his works. Whether or not mathematics is identified with the forms or in a world, just below them, does not alter Plato's identification of mathematical knowledge as a prerequisite of understanding sensible experiences. This resembles at least, Kant's view that experience of the physical world depends logically upon the categorising framework, of which 'space and time' are critical constituents, for it to be comprehended. Now, for Plato, 'astronomy' depends upon mathematics, just as for Kant, 'Newtonian mechanics' was to be seen as depending upon mathematics. However, mathematics is to be taken here, in the sense of 'pure natural science', by which one can explain particular empirical situations, rather than as a fundamental category.[1]

That there is this similarity between Plato and Kant ought not to give the impression that the closeness covers all their views of mathematics. Kant follows Aristotle in denying that mathematical objects have ontological reality. One can recognise that Kant's immediate answer to the question, 'Are there eternal objects of mathematics?' will be that mathematics consists of the 'conceivable' and not the 'perceivable'. The 'perceivable' is dependent upon the world, and the 'conceivable' upon man's nature. While Kant believed that Euclidean geometry was the only way to make sense

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1. Compare for example, Plato, Laws, Book VII, pp. 818ff., with Kant, Critique of Pure Reason, I, First Division, Chapter II, Section 3. Kant distinguishes 'pure natural science', which depends logically upon synthetic *a priori* principles, from other areas like chemistry, which are totally synthetic *a posteriori*, in their foundations. This striving for absolutes indicates the extent to which Kant remains tied to Rationalist predecessors, rather than to any relativism that evolved in the two centuries after his death.

of space as man experiences it, he did not deny the logical possibility of other geometries, any more than he would deny the conceivability of 'actual infinity'. Kant limited this, to the belief that, 'we can neither perceive nor construct an actually infinite aggregate' (Koerner quotes Kant, p. 30). However, Kant suggests a slightly tighter notion of the limits of mathematics, by requiring that the entities are not to be just 'conceivable', but 'constructible' in the mind, according to the rules of mathematics. It is this linking of the objects of mathematics to what can be constructed that indicates the justifiable claim that Kant is the father of Intuitionism.[1] In such a theory one must ask two questions, 'Is it conceivable?' and then, 'Is it constructible?'. In this context, 'actual infinity' is conceivable but not constructible. Kant criticises Aristotle for not recognising that proof that 'actual infinity' is not constructible is not to make the phrase meaningless, nor establish that God could not construct it. It is worth repeating that 'constructible in the mind' is the sign of 'ideas produced' rather than the uncovering of objects in a physical or transcendental reality.

The kinds of responses that Plato, Aristotle, Leibniz and Kant would make to the first question, 'Are there eternal mathematical objects?' have been considered, and further understanding may be facilitated by considering their likely responses to two secondary questions.

Firstly, they would all reply positively to the question, 'Is there mathematics?', and would also respond positively to the question, 'Is there eternal mathematics?' where the interpretation is that changes in empirical states of affairs will not alter mathematics *per se*. However the responses to the initial question are split on ontological grounds. Plato responds without reservations that there are 'eternal mathematical

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1. See the detailed discussion of 'intuitionism' in Chapter 3.

objects'. The others refrain from the connotations of 'objects' as 'extensional' and 'concrete'. Aristotle, Leibniz and Kant all reject the strong form of 'Platonic realism', but their alternatives will not become fully clear until the epistemological question, 'what is identified by the phrase "mathematical truth"?' is considered at length, in Chapter 5. What has been resolved is that 'ontological realism' is one distinct position and that there are others like Idealism and Nominalism as well as less extreme types of realism, to be seen in more developed forms later.

QUESTION TWO: '*Is "mathematics" logically distinct from the "empirical sciences"*?'

The united reply on this occasion is, 'Yes', BUT with qualifications, particularly by Aristotle. This means that a consensus position cannot be presented but again each philosopher will be considered separately.

Aristotle draws upon the special distinction between 'existing potentially' and 'existing in actuality'. All 'things that exist potentially are discovered by being brought to actuality' (*Metaphysics*, Book IX, Chapter 9). This 'existing potentially' is found in man's thinking. Thus one sees a system of 'hypothesis' and 'deductive proof' applicable to all science. Whether the hypothesis is,

'three angles of a triangle add up to two right angles'

or 'these bulbs planted in this soil will produce tulips',

does not indicate a difference, nor would the deductive proof based upon definitions indicate a difference ('intuitively' established in O'Connor's view, but this is open to interpretation as has been indicated earlier). However, there is a logical difference in the construction 'in actuality'. The 'actuality' of a mathematical example may rest at construction in 'thinking' which is itself actuality.[1] The 'actuality' of a scientific example is so, not only in 'thinking' but also necessarily in the construction of 'substances', in this case, 'tulips'. However, the fact that the 'tulips' grow is not the proof of the hypothesis for that is verified, like the mathematical one, through the internal coherence of the particular scientific system (this will be returned to in Chapter 5). To make this last point clearer, the failure of the tulips to grow would not be counted as disproof, for given the coherent proof, they 'ought' to grow.

Thus for Aristotle mathematics is separable logically both from all other sciences and from formal logic. Formal logic provides 'the principles which are common to all sciences' (Koerner, p. 20), but is not sufficient to establish all mathematics for which its own special principles and definitions are necessary. Mathematics is separable from all other sciences because of the underlying ontological position that tulips exist and it is necessarily the case that 'five' and 'seven' do not.

Like Aristotle, Kant asserts the logical discreteness of mathematics from both formal logic and the empirical sciences. The most ready way in which one remembers this discreteness claim is in Kant's introduction of 'synthetic *a priori*' for propositions describing universal particulars, as mathematical propositions do. Formal logic is analytic *a priori*' and empirical sciences consist of 'synthetic *a posteriori*' propositions, consequent upon synthetic *a priori* principles. Aristotle may be interpreted as coming towards such a distinction, because he refuses to identify mathematics with logic (as Leibniz does), or assert an 'absolute

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1. In *The Concept of Mind* (pp. 143-7) and in the article, 'Teaching and Training' (pp. 109-10, in *The Concept of Education*), Ryle indicates his debt to Aristotle for the classification of 'achievement' verbs as distinct from other verbs of activity or process. Furthermore, cognitive verbs like 'thinking' and 'deducing' are identified by Ryle as resembling 'achievement verbs' but not being members of that class. This is directly supportive of Aristotle's separation of them, as 'actuality verbs'. Another verb of this kind would seem to be 'teaching' for it too can be used with an in-built success condition, ('You taught only if he learned').

reality' of mathematical objects (as Plato does). As will be noted below, Leibniz and Plato have no reason to separate mathematics from logic, but Kant is concerned to assert the special relationship between mathematics and science, that is, to see the framework of all science as mathematics, and the framework of all thinking, including mathematical, and hence scientific, as logic. Pure mathematics provides the means by which man orders his physical world, according to the constraints of the dimensions of 'space' and 'time'.[1] This ordered experience of the world is not to be confused with the specific study of this world, which is called 'science'.

Mathematics is not an empirical science for it is a pre-condition of empirical science itself. Nor is it just a set of analytic propositions (tautologies)[2] because mathematical propositions themselves act as participant elements in the description of man's physical experiences. Mathematics is recognisably distinct therefore.

For Leibniz, the separation of mathematics from the empirical sciences is based largely upon the distinction between 'Truths of Reason' and 'Truths of Fact'.[3] For Leibniz, anyone who understands the definition of '5' and '7' must be contradicting himself necessarily if he denies, '5 + 7 = 12'. The same man is not contradicting himself necessarily if he asserts, '5 apples and 7 apples do not make a dozen' or 'salt is not always soluble in water'. It is logically possible that what is generally accepted as a

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1. 'Space' itself is describable only because there is a Euclidean geometry to provide the vocabulary and 'time' depends upon the sequences of arithmetic for its description. However the absoluteness of the description of 'space' and 'time' is not evidence, it is to be remembered, of the absoluteness of Euclidean geometry (See the earlier comment made on p.19 above).

2. To be just a set of 'analytic' propositions would make mathematics, logic. Analytic propositions are true because the meaning of one part of the proposition is included within the meaning of the other part. 'All bachelors are unmarried men' or 'all men are bipeds'.

3. 'Truths of Fact' cannot be unpacked simply by considering the definitions of 'the subject' and 'predicate' that constitute all Leibniz's propositions. For example, 'every judge is over fifty' or 'my friend, John is asleep' are propositions that require consideration of empirical evidence for their demonstration.

'truth of fact', even in applied mathematics, could be shown to have some other answer. As strongly as Leibniz separates mathematics from empirical science, he brings mathematics into a unity with logic. He believed that there must be one formal system in which all truths of logic and of mathematics would be decided. In such a system, any rational man, given the right rules and method, would be able to establish the truth of any logical or mathematical proposition. The programme is precisely that taken up by the Logicists, and the method of arithmetisation was that which Goedel was to use to demonstrate the impossibility of the programme.[1] As with the logicists, the distinctive terms 'logic' and 'mathematics' would cease to exist - 'logic and mathematics' is one form of knowledge.

While Leibniz saw the unification of science with 'logic and mathematics' as possible for God alone - only God can apprehend scientific

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1. From the age of 20, if not earlier, Leibniz uses numbers to indicate words and in particular, the terms of geometry. Rather like someone coming across Descartes' algebraic representation of Euclidean geometry and attempting to simplify it further, he reduces the theorems to arithmetic relationships that a machine could carry out. In 1679, at the age of 33, Leibniz wrote a paper entitled, 'Rules from which a decision can be made, by means of numbers, about the validity of inferences and about the forms and moods of categorical syllogisms' (Leibniz: Logical Papers, p. 25). Goedel provides a proof in 1934 by using the arithmetisation of Z (whose elements are the non-negative integers) that there is a statement such that neither the statement nor its negation is provable within Z. The point of relevance to the present discussion is not Goedel's theorem but his method which is to employ the possibility of arithmetising formal systems to prove something about statements within the system itself. Put very crudely, the idea common to Leibniz and to Goedel is that one might link concepts to numbers and by manipulating the numbers discover the 'truth or falsity' of the proposition which the concepts constitute. Thus Leibniz builds up an arithmetisation of geometric terms as in the following example (ibid., pp. 6-7): If 'Space' is 2; 'Between' is 3; 'Whole' is 10; then 'Interval' is 2.3.10. An 'interval' is the whole space between (ibid., p. 7). Similarly, 'Any term of any proposition,...is to be written as two numbers...For example, let the proposition be "Every wise man is pious", and the number corresponding to "pious", + 10 - 3...the two numbers of the same term must not have a common divisor...A true universal affirmative proposition, for example

   Every wise man is pious
   +70 -33 +10 -3

   ...is one in which any symbolic number of the subject (e.g. +70 -33) can be divided exactly...by the symbolic number of the same sign belonging to the predicate.' (ibid., p. 26).

definitions that involve an infinite number of components - Plato seems to have greater faith in 'educated man'. In the Republic, 'dialectics' and 'mathematics' are equally dependent upon the reasoned experience of their absolute reality, 'the Forms'. That Plato did not extend this theory explicitly to astronomy until later works like the Laws, is due more to the predominant position of mathematics in Plato's time, than to any personal indifference for astronomy.[1] For Plato there is no logical separation of 'knowledge' for it all exists in the Forms, whether science, mathematics, logic or even morals. The earthly world approximates to the transcendental reality and the question to be asked in a problem about the earthly world is, 'To which principle in the Forms does this problem approximate?' For example, two apples and two apples' approximates to '2 + 2' which make '4', and so, 'two apples and two apples make four apples'. Thus for Plato, there is absolute knowledge in 'Pure Mathematics' and 'Pure Science' as a unified area, to which correspond the approximate systems of 'Applied Mathematics' and 'Applied Science'.

The answers to the second question seem to fit into two kinds. On one side, Aristotle and Kant assert the separation of Mathematics from both Logic and Science, and on the other side, Plato and Leibniz assert the ideal situation in which Logic, Mathematics and Science are unified, but Leibniz admits that necessarily given that man is limited, Science is separate from the unified system of 'Logic and Mathematics'.

CONCLUSION: In this chapter the bases of various philosophies of mathmatics are laid down. Plato and Leibniz point one unquestionably to Logicism, while the concern of Aristotle and Kant for 'constructibility'

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1. See the support given to this in A. E. Taylor's Plato, pp. 292ff. and pp. 497ff. For example, p. 293 'the "reduction of all pure mathematics to logic" is only a part, and not the most important part, of what the Republic understands by "dialectic". Such a unification of the sciences as the Republic contemplates would require a combination of the reduction of mathematics to logic with the Cartesian reduction of the natural sciences to geometry.'

shows the path to Intuitionism. In considering these philosophies and others, in greater detail, it will be further indicated how such dependence arises.

CHAPTER 2

LOGICISM AND FORMALISM

INTRODUCTION

With the work of Frege, philosophical interests in the late nineteenth century returned to questions of logic rather than to epistemology. Aristotle had been the focus of philosophical debate throughout the middle ages, until Descartes and others put Aristotle's logic to one side and argued that epistemological questions were more fundamental. Descartes had been firstly concerned to decide what, if anything, one 'knows', and then concerned with the logical nature of arguments by which one demonstrates that what one 'knows' is true. With the possible exception of Leibniz, no philosopher attempted to provide a replacement for Aristotelian logic for another two hundred years, until followers of Kant, e.g. Fichte and Hegel rejected it, in favour of a dialectical approach. The nature of their rejection need not bother us here, as it had no immediate consequences for the philosophy of mathematics, and does not seem to have made the climate any more receptive to Frege's modifications of Aristotelian logic. The stimulus for Frege to produce a richer logic did not come from philosophy, but from the range of advances just achieved in mathematics. Frege introduced a logic with universal quantifiers, the propositional logic,[1] which would be a

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1. By 'propositional logic' Frege attempts the symbolic representation of propositions not covered by the syllogism-dominated logic of the Aristotelians. This logic had been almost exclusively concerned with propositions that could be rewritten as one subject with one predicate. A typical proposition of this elementary predicate calculus would be, 'Socrates is bald'. For Frege a necessary part of any useful logic would be that predication can occur over several variables - 'Socrates is balder than Plato' or '$x > y$'. In this feature Frege returned to
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sufficiently powerful tool to give rigour to the work of Cantor and others (for example, in the study of transfinite numbers). So it was, that Frege attempted to provide what he believed would be the strongest possible support for arithmetic and numerical analysis,[1] its reduction to Logic. This is the essential feature of Logicism.

Twenty-five years later, Hilbert was able to reflect upon the work of Frege and upon challenges to it by Kronecker and Cantor. Thus he could set out to produce a theory, free from the paradoxes of set theory.[2] In his lecture, 'Foundations of Logic and Arithmetic', he says,

> Arithmetic is often considered to be a part of logic, and the traditional fundamental logical notions are usually presupposed when it is a question of establishing a foundation for arithmetic. If we observe attentively, however, we realize that in the traditional exposition of the laws of logic certain fundamental arithmetic notions are already used, for example, the notion of set and, to some extent, also that of number. Thus we find ourselves turning in a circle, and that is why a partly simultaneous development of the laws of logic and of arithmetic is required if paradoxes are to be avoided. (Heijenoort, p. 131).

Fn. 1, p. 27 contd.
developments already indicated in Leibniz's work. However, Frege distinguishes the content of a proposition from the judgment or assertion about the proposition, i.e. distinguishing '-p' from '⊢p', 'the meaning of "Socrates is bald"' distinguished from 'the assertion that Socrates is bald'. Previously, the two senses had been conflated and it was assumed that each proposition was asserted, while in the following argument given in Frege's revised calculus, p is only asserted in the conclusion: ' ⊢(not m), ⊢(m or p), ergo ⊢p'. For Frege, each assertion has a reference, and so the assertion sign reduces the unnecessary multiplication of objects. It also separates the description of the pieces in the calculus-game from action within the game, i.e. 'm' is the description of a piece like 'pawn' in chess, while ' ⊢m' is like a move in chess, say 'pawn to K4'. Previous attempts to unseat Aristotle include the works of Occam and Boëthius but not until the nineteenth century was there widespread dissatisfaction.

1. Frege accepted Kant's view of 'geometry' as synthetic *a priori* and not reducible to logic. Whitehead intended to attempt the reduction but never did. Hilbert completed an axiomatisation of geometry which was sufficient for the Formalist programme, as there was no wish to reduce this to logic, but Hilbert later argued that the formal system of geometry must be totally subsumable within an axiomatisation of arithmetic.

2. The most famous paradox of set theory is that found by Russell, 'Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows.' (From 'Russell's Letter to Frege', Heijenoort, p. 125).

Here Hilbert is establishing the distinctive programme of Formalism, as against that of Logicism, as 'the conceptions and means of investigation prevalent in logic,..., do not measure up to the rigorous demands that set theory imposes'. Hilbert asserts the separate axiomatisation of logic and of mathematics, because logic is not in itself sufficiently rigorous to be formally decidable and could not encompass mathematics in any case without the primary insertions of basic arithmetic terms,i.e. these terms are not definable in the terms of logic alone. Formalism attempts the demonstration that mathematics can be seen as a complete and consistent[1] system of interrelated formulae. Similarly logic would be another such system and so on.

By the first decade of the twentieth century two rival movements had been established. It is the replies that they would give to the two questions discussed in Chapter 1 that will provide the core to the remainder of this chapter.

<u>QUESTION ONE</u>: <u>Are there eternal Objects of Mathematics</u>?

<u>Frege</u> asserts unequivocably that the proper explanation of the ontology of mathematics is to be platonic. The objects of mathematics are not empirical, nor mental, but eternal. Furthermore, no asserted proposition may leave any ambiguity as to which object reference is being made. Thus, $\sqrt{4} = \pm 2$ is logically unacceptable as an asserted statement, for one does not know if the reference is +2 or -2. Similarly one must never fail to make restrictions as to division by zero.[2] Thus

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1. By 'complete' one means that every statement within the system is either provable or disprovable within the system. By 'consistent' one means that there are no pair of statements in the system which are both provable and contradict each other.

2. Frege bases his analysis of logic and the logic of language upon the kinds of relationships found in mathematics. Central to mathematics is the relation, 'function' and 'argument'. A function, '$y = x + 3$' only identifies particular numbers, the 'objects', when an argument, a value of x, is put into the function. Frege believed that the logic of language, and the logic of mathematics, could be formalised by the use of a similar relation, that of the 'concept'. Thus the
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the logicist follower of Frege would posit an arbitrary value for the result of dividing by 'x - y' when 'x = y'.[1] Furthermore, all symbols must be formally introduced into the system, so that ambiguity is eliminated. Compare,

1) $x + 3 = 7$; 2) $\xi + 3 = 7$; 3) $\square + 3 = 7$.

In 1) x is an indefinite placeholder for a numeral and defined as such within the logic. In 2) ξ is an indefinite placeholder for anything including numerical signs, but $\square$ in 3) has no meaning at all within the logic. Technically it requires formal definition, before the resulting activity that may occur in a Primary School, say, can be judged mathematics. Thus the pupils' attention must be brought at some point to this requirement if they are to be initiated into mathematics, as described by Frege.

Hilbert accepted that within finite arithmetic there are real objects to be employed in mathematics. These would be empirical strokes, not platonic forms. However, mathematics is not just a science of finite numbers and no one can see the strokes of an actual infinity. Thus one must add 'ideal entities' to make up the system. These entities would be essentially signs defined within the system, intuitively grasped, and without any kind of reference, neither earthly or eternal. In fact, there are no eternal objects of mathematics at all for it is synthetic a posteriori, just like any science. Its components are no more eternal than the electrons and cabbages that other scientists study. Hilbert was not concerned about the ontological question, provided he produced

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Fn. 2, p. 29, contd.

'open concept': 'players on a field for one team', only becomes closed when one replaces 'players' by 'cricketers'. Furthermore, it has the numerical reference, '11'. Thus the specific concept has an object reference, as must any properly satisfied concept, and the object has ontological status whether it is a house, a number or a unicorn. Furthermore, any talk of 'knowing an object' depends logically upon being able to give an appropriate concept to which it applies, e.g. 'I know that this is Keele because it is the village in which I live'.

1. For example, the function '$y = \frac{2}{3 - x}$' is defined everywhere but at a point of discontinuity, '$x = 3$'. Following Frege one ought to give this some arbitrary value, say '$y = 0$', for it is axiomatic for him, that if there is no formal rule to exclude the particular value of 'x', then 'y', the function, must have a value.

an internally consistent axiomatisation. Hilbert adds all the notions of Cantor[1] to his system of mathematics, so long as they are not identifiable as the source of internal contradictions. If this programme of Hilbert's had been successful, then one would have known by mechanical[2] means whether or not a particular proposition is true or false. Thus for Hilbert, Formalism was to be seen as meaningful in so far as it provided proper foundations for mathematics. Hilbert was not concerned with formal systems in general, but with that formal, internally consistent system, which could be given an interpretation picking out the whole of mathematics. Cantor's paradise is only a paradise as against a modification of a given formal system, if one has given it a particular interpretation. In other words, one might invent some system in which 'x_n' is generated by '$K^x j$' and there is no way of deciding if '$j = n + 1$', but only that 'x_j precedes x_n', and this allows considerable manipulative moves, without any ontological consequences. The ontological consequences arise if one interprets 'x_a' as '$\aleph_0$' and 'x_b' as '$\aleph_1$', for then one wants to know if there is some transfinite number that lies between the transfinite number identifiable with all rational numbers, $\aleph_0$, and the

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1. Cantor produced a theory of transfinite numbers, $\aleph_0, \aleph_1, \ldots$, 'Cantor's paradise'. Hilbert saw this as just one more enrichment of the game - as someone else might see the addition of a third dimension to chess. As long as one can keep playing and one can find additional satisfaction, then there may be no reason for not having the extra dimension as an optional extra. This had been Leibniz's reaction to actual infinity two hundred years earlier, that it is a fiction, but being a serviceable fiction it may be included.

2. 'Mechanical' is a term used by all Formalists and is commented upon by Wittgenstein in his Remarks on the Foundations of Mathematics most interestingly. He makes the point that the analogy is rather an odd one, for mathematics is in fact claimed to have a level of certainty better than any known machine. Machines are liable to go wrong, but the Formalists see their systems as being those with the least imaginable risk of error. The danger that Wittgenstein is highlighting for the Formalist is that implicitly he seems to assume that the systems are 'error-free', analogous to 'ideal machines', while for logical consistency the Formalist only claims, as a matter of fact, a small possibility of error. To make any stronger claim would be to turn 'the manipulation of strokes' into an ideal state, certainly contrary to Strict Formalism (see Klenk, Wittgenstein's Philosophy of Mathematics, p. 32 for a fuller explanation).

and the transfinite number identifiable with all real numbers, $_1$, according to the axiomatisation, that does not appear in transfinite arithmetic itself: Cantor's paradise. Hilbert's followers were to take up a stricter notion of Formalism. Their notion was not tied to particular interpretations of already produced, symbolic systems. Curry and others who followed Hilbert saw 'the science of formal systems' as providing axiomatisation of any formal system, whether consistent or not, but including 'mathematics'. To these strict formalists Cantor's paradise was not seen as a problem, for people like them studied formal systems independently of particular interpretations. However there was a problem for someone who wanted to make use of the system, and found inconsistencies arising in the actual application. Thus, on the one hand Hilbert had had a priority: the axiomatisation of a mathematics with the maximum richness and it was this desire that motivated his famous cry, 'No one shall drive us out of the paradise which Cantor has created for us.' ('On the Infinite' in <u>Philosophy of Mathematics</u>, p. 141). On the other hand, the strict formalist would have far less fear of such an eviction, for this would come from a philosophical argument about the inadequacy of the given interpretation, and that should not arise in a non-interpretative science. Furthermore, they could argue that there is no logical reason to stop the study of a system that is shown to be inconsistent, for one cannot know <u>a priori</u> that only consistent systems will have interpretations that can be of use to another science.

Hilbert was only interested in the axiomatisation of systems that already had some kind of 'meaning'. Interpretations are therefore a key feature of Hilbert's Formalist programme, while linking interpretations to formal systems has no part in the science that the strict Formalists identified. These Formalists would proliferate games, in order to study their behaviour. They agreed with Hilbert that these were not eternal objects, but exist in the same reality as everything else studied by scientists, the world of phenomena.[1] The formalist definitions of mathematics may be summarised in two phrases. Hilbert would require the notion,'the science of formally consistent systems', while the later formalists would stop at 'the science of formal systems' and 'regard to consistency is thus no part of the formalist conception of mathematics' (Curry, 'Remarks on the Definition and Nature of Mathematics' in Philosophy of Mathematics, pp. 155-6.

Russell rejects both this corrigible view of mathematics and also Frege's platonic conception. He does accept Frege's logicism in so far as it is a programme to attempt the reduction of mathematics to logic, but he can make no more sense of 'eternal objects' than did many contemporaries of Plato, of his forms. Russell's logicism is one founded on 'classes' rather than 'objects', that is, the definition of numbers[2] rather than any claims to the existence of numbers. One is reminded of the distinction drawn on p.18 above between Aristotle's 'idealizing abstraction' and Platonic 'forms'. Russell takes the views of Plato and of Aristotle on a further stage. He is forced by the paradoxes of set theory to produce a hierarchy of types. Thus the predicate described in footnote 2, p. 28 (the self-predicating predicate) is admissible but the negative answer to the question is stipulative or is simply not admissible, by stipulation. The point being that a rule is introduced which is not to be found in classical logic and is defined so that a predicate cannot be predicated of another of the same 'type'. The other 'artificial' rule introduced by Russell was the 'axiom of reducibility' which brings back

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1. 'Phenomena' is taken here in the sense of 'sense-data' as used by Russell and Positivists.

2. Russell also highlights the distinction between numerals and numbers. Numerals are the names given to numbers and can be used outside mathematics, e.g. 'Is "2 + 2 = 4" a true mathematical proposition?' may be asked of a mathematician to open up a social conversation without any intention on either part for the resultant conversation to be mathematical. However, if one is asked to consider 'the factors of 12' then what one studies is not a numeral but a number, and this could only occur within a mathematical context.

down the number of predicates proliferated by the theory of types.[1] The result for Russell is a world of discerning terms without the guarantee that the terms refer to any existent objects; rather as the phrase 'present king of France' may have no existent reference, but a possibility of usage in language. One may use predicates successfully to convey meaning, without attaching them to the real world - novelists do this all the time. Such are the predicates of logic and mathematics. They never identify objects in the world. For Russell, a question about the ontology of mathematics indicates ignorance of what mathematics is. Whenever one identifies something that seems to be an object of mathematics, like 'the first prime number after 2' then either it is a contextual definition or is reducible to such. In other words, nothing 'out there'[2] is ever discussed within logic or mathematics - just as 'Mr. Silly' has no meaning outside of 'Mr. Men Books', its only possible context.

Russell took this position because he recognised it as the most consistent, if one accepts that logic has no particular objects. Logic is a framework for, and not a description of any reality. Thus, if mathematics is reducible to logic, then, something cannot come out of

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1. The theory of types may be exemplified as follows: Let us imagine a system in which Football Teams are 'individuals' of 'Type 0', and predicates of the form 'beaten by/drew with/beat x on date y' are of 'Type 1', and predicates of the form 'coming nth. in year k in division p' are of 'Type 2'. The theory of types dictates that predicates of type 2 cover predicates of type 1 and so on, and cannot themselves involve other predicates of type 2. In this example, to fill out a type 2 predicate so that it is a proposition will involve links with a set of type 1 predicates - the football results for a given season. Using this example, the axiom of reducibility may be seen as follows: Take 'Bristol Rovers' as a Type 0 - individual and 'beaten by West Ham on 17th March' as a 'Type 1' predicate, then both have the same reference and so may be treated as indiscernible, given the axiom of reducibility - only one 'nomen'.

2. Wittgenstein came to argue that replacing a description by a name allows one to 'create' new things, and so the name is not just a substitution as Russell thought, but facilitates development, and in that sense it comes to have a life of its own, e.g. 2^7 can always be rewritten as 2x2x2x2x2x2x2, but once one's notation includes $2^a \times 2^b = 2^{a+b}$, then $2^7 \times 2^{19} = 2^{26}$ is 'perspicuous' but not $2x2x2x2x2x2x2 \times 2^{19} = 2^{26}$. See <u>Remarks on the Foundations of Mathematics</u>, pp. 65-90.

nothing, there can be no mathematical objects either. This view of Russell's is called 'nominalistic logicism', and Frege's answer to posit objects for logic too, is called 'realistic logicism'.

While there are many contemporary views, strongly influenced by Logicism, they accept the Logicist thesis to differing extents. While Carnap retains the idea that arithmetic is reducible to logic in his 'constructional system', Quine only accepts that there is a model of 'arithmetic' identifiable in 'set theory', and that consequently one can 'get by without numbers' (Compare Carnap's article, 'The Logicist Foundations of Mathematics' with Quine's articles, 'On What There is' and 'Two Dogmas of Empiricism'; reprinted together in Philosophy of Mathematics, pp. 31-41; 183-96; and 346-65). Quine's is a logically weaker programme than that of Frege and Russell, for Quine accepts links with direct experience on Pragmatic grounds (in particular, on the grounds of Occam's razor, not to have more entities than is absolutely necessary). He argues that if sets can do everything that numbers can do, then one does not need to consider whether numbers exist or not. Quine has two principles at the centre of his thinking for all 'science, mathematical and natural and human' ('Two Dogmas of Empiricism') and one of these is Occam's razor.[1] By Occam's razor Quine argues that a set theoretic foundation is preferable to an arithmetic one, for the whole of mathematics, because the laws of set theory cover all those of arithmetic. Therefore, choosing set theory as the foundation reduces the total number of necessary laws in the universe. Put another way, mathematics has to have some ontological foundations, but one has a free choice between 'arithmetic' and 'set theory' except that the latter provides ontological foundations for other systems

1. The other principle is that all theories 'must be kept squared with experience' ('Two Dogmas of Empiricism'), and is consistent with Quine's rejection of the analytic/synthetic distinction, discussed in Chapter 5 below. It also indicates the pragmatist viewpoint that everything must 'square with experience'.

besides mathematics. Thus one can see that Quine's pragmatic view leaves the line far less clear between Formalists and Logicists.

To sum up the replies to question one: the Formalists all reject 'eternal mathematical objects' but Hilbert considered some mathematical objects to be 'ideal entities'. The strict Formalist treats all mathematical objects as empirical constructs, 'strokes on the page'. The Logicists on the other hand, are fundamentally divided between a 'platonic reality' and 'nominalism'. The platonists like Frege believed in eternal 'entities in a realm accessible only for thought' (Complementarity in Mathematics, p. 106), while the nominalists like Russell believe that the entities have no independent status, but have meaning within propositions within a system.[1] The division between realism and nominalism is not altered by the weakening of the Logicist programme in the way Quine does - 'sets' can be still either 'eternal objects' or 'verbal descriptions' - and this seems to remain unresolved.

QUESTION TWO: Is 'mathematics' logically distinct from the 'empirical sciences'?

The Logicists, Frege and Russell, would assert that 'mathematics' is logically distinct from the 'empirical sciences', for it is logically reducible to 'logic'. 'Logic' consists of analytic *a priori* propositions while the 'empirical sciences' consist of synthetic *a posteriori* propositions. Thus there is no logical possibility of overlap. Frege refers to the distinction drawn by Leibniz between 'Truths of Reason' and 'Truths of Fact' (see p. 23 above), rather than to Kant's analytic/synthetic distinction. The point of the argument is the same whatever the terminology. A third kind of dichotomy, besides 'Reason and Fact' or 'analytic and

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1. Frege and Russell differ more in degrees than in quality. Frege also recognised that a concept only has meaning in a proposition but asserts that the concept and the proposition necessarily have references in objects of a platonic reality. It is the necessity of the connection that Russell questions, for Russell rejects the argument that every concept and every proposition must carry existential importance.

synthetic' is provided by Quine in his work (see for example in 'Two Dogmas of Empiricism' and in Word and Object). Quine rejects the idea that one can find a reference for the phrase 'analytic proposition'. He argues that there are no 'purely' analytic propositions, for all propositions, even if only by inference, are open to modification because of sense experiences.[1] There does seem to evolve in Quine's theory a dichotomy between 'peripheral sentences' that are directly open to confirmation by sense experience, and 'non-peripheral sentences' that are only open to revision by inference. Thus Quine separates 'logic and mathematics' from 'empirical sciences', because the latter alone include 'peripheral sentences'. In other words, one corrects 'logic and mathematics' when the framework it provides for the 'empirical sciences' is inferred as leading to false assertions about peripheral sentences, but 'logic and mathematics' make no direct assertions about the peripheral sentences.

The result is that even Quine's form of Logicism leads to the assertion that mathematics is separable from the empirical sciences but Quine might hesitate in calling the separation 'logically distinct'.[2]

The Formalists would be united in rejecting a logical distinction between mathematics and the empirical sciences. They both consist of

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1. The sense in which one might use the word 'analytic' of 'there are 7 days in a week' is that of a learned stimulus-response rather in a Humean fashion. Thus one responds immediately to the question, 'how many days in a week?'. There is a psychological tie and great confusion would result from its denial, but conventions of this kind could alter. The implications of this position for notions of 'truth' will be explained in Chapter 5 below, but it may be worth noting here that the axioms of set theory are also examples of 'analytic' sentences.

2. Quine's uncomfortable feeling would be based on his recognition of the fundamentally social use of language, under which the boundaries of mathematics could be otherwise than as they are. Quine is most appropriately seen as following in the Pragmatist tradition of James and Dewey, rather than as many writers in the philosophy of mathematics describe him, as a pragmatic logicist. Quine himself keeps clear of identification with logicism, which can be taken to indicate the extent to which Quine has abandoned the original programmes of Frege and Russell.

synthetic a posteriori propositions, and to use Quine's expression, of 'peripheral sentences'. The strokes manipulated in mathematics are as open to sense experience as the chemicals analysed in a laboratory. Hilbert saw the mathematician as an investigatory scientist who investigates the relationships among numerals, etc. Every science consists of the objective study of some delimited range of entities and for mathematics this range is 'complete and consistent systems'. Hilbert's orientation reminds one of Locke's reference to mathematics as the handmaiden of science. Unlike his later followers, Hilbert selected the systems for axiomatisation on the basis of their known service to other sciences. The strict formalists like Curry make no such division, and Goedel's theorems[1] demonstrated that Hilbert's talk of completely axiomatising arithmetic was hopeless. Curry and others have carried on, citing the belief that even inconsistent formal systems may have useful applications. The result is a Formalism committed to the science of all formal systems.

Wittgenstein argued (Remarks on the Foundations of Mathematics, IV, 1-12) that this Formalism makes mathematics indistinguishable from all games, that 'the science' presented is asserted as having no meaning and no interpretation, which seems to contradict one's basic conceptions of

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1. Reference was made to one of Goedel's theorems in footnote 1, p. 24 above. The theorem referred to there, demonstrated that Hilbert would never be able to axiomatise arithmetic so that every statement can be decided within the formal system. Goedel showed that there will be always at least one statement not refutable within the system. In other words, Hilbert's programme could not include 'complete decidability' - proofs could not be produced in toto 'mechanically'. The further consequence of Goedel's theorems is that the paradoxes noted by Russell and others cannot be guaranteed elimination by any particular rule(s), as Russell had hoped to achieve by the Theory of Types. Essentially, the further interpretation of Goedel's work is that paradoxes arise, if one attempts to define 'truth' in a formal system FOR that same system - this massive 'self-predication', as must occur in ordinary language, generates paradoxes. The Logicist is thereby left hanging on the horns of a dilemma - either modify the programme or accept piecemeal isolation of paradoxes. This area will be developed in the light of Tarski's work in Chapter 5 below.

a science. It is more like a game of patience whose successful completion has no implication for the world, rather than like the successful isolation of a rare virus - more readily typifying science. The Formalists seem to accept Wittgenstein's analogy and would broaden 'science' to include 'games and the study of them'. For the Formalists the successful proof of a theorem of set theory is much more like a game than any scientific study, which is specifically orientated to solve a problem in the real world. However, they may still argue that that is only one branch of science, and that there are others in which no interpretations are made.[1] Here one devises and studies 'games' for their own sake, while not dismissing the right of others to give them interpretations. There is still some ambiguity as to the use of the term 'mathematics' in the work of the strict Formalists. What seems to happen is that where mathematics once stood, one puts the phrase 'science of formal systems'. This was not a problem initially, for Hilbert seemed to consider the set of Formal Systems and Logic and Mathematics to be coincident. For the strict formalist, 'old mathematics' with consistent systems only, is a subset of 'new mathematics' consisting of <u>all</u> formal systems, and so they have a wider notion of mathematics. Also, Hilbert was concerned about interpretation and mathematics, as mentioned already on p. 33, and this is ruthlessly removed from the contemporary Formalist programme.

Formalists do seem to have stretched the notion of 'science', for ordinary language users do expect a science to be predictive of states of affairs in the real world, and the uninterpreted formal systems do not do this necessarily. Mathematics as a formal system may be empirical and may be investigatory, but its detachment from scientific prediction

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1. In part 2, Chapters 6 and 7, and in part 3 there is further discussion of the consequences of shying from 'meaning'. Crudely one could argue from a notion of 'meaning is use': that 'meaning' necessarily exists <u>within</u> a game or any formal system.

leaves it a 'peculiar science'. For the Formalists, as was indicated in the previous paragraph, it is essentially an extrinsic matter whether or not a formal sentence like, '1 + 1 = 2' is ever interpreted in say, the gathering of apples. This formal system will be used because it works, and only as long as it is seen to work.[1]

The answers to the second question divide in line with the views of the two differing movements. The Logicists, reminiscent of Leibniz, assert a unification of Logic and Mathematics, but leave Science logically distinct. The Formalists assert a position that did not arise in Chapter 1. They see Mathematics as one of the sciences where all sciences are _a posteriori_. Mathematics is discrete, not because of a peculiar method, but because the object of study is 'formal systems' rather than 'physical bodies' or 'cultures' or what-have-you.

Conclusion

In this chapter two movements have been identified and their programmes outlined in the light of the ontological and discreteness questions. It has been noted that the founding fathers of each movement were severely unsettled by discoveries occurring while they attempted to implement their programmes. The result of these difficulties has been the reformulation of the programmes by followers, at the cost of some of the original intentions of the founding fathers. Thus Quine's 'pragmatic logicism' is a programme of 'replacing' numbers by sets, rather than one of 'reducing' numbers to sets, and Curry's 'strict formalism' is a science of 'all' formal systems and not just consistent ones. Frege and Russell, one might suppose, would have come to accept that mathematics cannot be derived from logic alone and similarly, Hilbert would have come to accept

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1. This is not a satisfactory answer and will be discussed further in part 2, Chapter 8. The implications of a Game-orientation for the teaching of mathematics may be seen as restrictive of the study of mathematics in application, and will be contrasted in part 2 with other positions that link 'interpretation' to the very nature of 'mathematics'.

that mathematics cannot be completely axiomatised so that the 'truth' of any proposition can be proved 'mechanically'.

Still there remains opposition between contemporary Logicists and Formalists, both on the nature of mathematical objects and on the relationship of mathematics to other subjects. The leaning on one side to 'Truths of Fact', has not been eradicated, for all the changes of the past fifty years. However, during this time, Kant's synthesis of this dichotomy has been taken up by a movement known as either 'Constructivists' or more generally, as 'Intuitionists'. In focusing on this movement in the next chapter, particular attention will be paid to the contrasting ontology presented there, as against those found in this chapter.

CHAPTER 3

INTUITIONISM

INTRODUCTION

A French contemporary of Frege, Poincaré was influenced by Kant just as much as Frege was influenced by Leibniz. The result was that Poincaré opposed the Logicist programme from its very inception. He argued that mathematics could not be reduced to logic because the potentially infinite has no basis in syllogistic reasoning. In other words, he believed that Frege's propositional calculus (see p. 27 above) was a corruption of classical logic. Poincaré was claiming that this programme, involving the inclusion of universal quantifiers and mathematical induction, would founder on circularity.[1]

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1. The thread of Poincaré's argument (Science and Method, pp. 143-96; Science and Hypothesis, pp. 1-16) can be explained as follows: 'the proposition, 'all jam tarts are delicious' can be demonstrated as true, only if one assumes an equivalence between this proposition and, 'this jam tart is delicious and this jam tart is delicious and...' where the truth of a finite number of particular propositions is taken as sufficient proof of the truth of a universal proposition. The model for this regularity, so Poincaré argues, is mathematical induction where proving a formula true for n = 1 and n = n + 1 (assuming it true for n) is taken as proof that it will be true for any n. In so far as classical logic is strictly finitist, one could check in principle the truth of any proposition, but mathematics requires consideration of at least, the potentially infinite, and so the logicist reduction is fundamentally absurd. This reaction can only begin, if one implicitly assumes the principle described by mathematical induction, as a logical axiom. It cannot then be claimed that mathematical induction is produced out of the logic. Hilbert, the founder of Formalism, recognises in his Foundations of Mathematics, 1928, that Poincaré's arguments are directed at Formalism as well as at Logicism. The modified argument is that mathematical induction cannot be justified within a formal system, for a pair of propositions may each be demonstrated by mathematical induction and yet contradict each other. There is no further means to indicate the resolution of the contradiction. Thus a system that includes mathematical induction cannot be shown to be consistent. Poincaré believed that mathematical

[Contd. overleaf

No movement grew up around Poincaré who was much more concerned with practising mathematics than founding a philosophy of mathematics, but in the two decades following his renouncing of Logicism a man did attempt to provide and implement a contrasting programme to that of the Logicists and Formalists. This man was Brouwer and twenty years after Hilbert's formalist foundations for arithmetic came Brouwer's intuitionist foundations (roughly during the period 1904-1926). Brouwer presented a more radical programme than any considered by Poincaré. Brouwer rejected both Frege's calculus and the classical logic of which it is an extension, in so far as these logics are presented as adequate modes for describing reasoning involving infinite species.[1]

Brouwer was influenced by Kant as well as Poincaré but unlike either of them, claimed that intuitionist mathematics is 'an essentially languageless activity of the mind, having its origin in the perceptive of a move of time'. Kant had placed mathematics firmly in the mind but had linked it to the 'perceptive' of space as well as time.[2] Brouwer retains the link with 'time' alone. He argues that mathematical activity must not be confused with the linguistic description of the activity. This description has rules, those of classical logic or an extension of it, but mathematics in itself is autonomous, independent of any given logic. Mathematics shapes its own logic as it is produced.

<u>Fn. 1, p. 42 contd</u>.
induction was a synthetic <u>a priori</u> principle rather than analytic (Logicism) or empirical (Formalism), for it 'is necessarily imposed on us, because it is only the affirmation of a property of the mind itself'. (Quoted by Kuyk, <u>Complementarity in Mathematics</u>, p. 112)

1. 'The set' has no direct equivalent within intuitionism. 'Sets' are passive collections in classical mathematics, while 'species' are open collections picked out by the assignment of a single characteristic property. Compare 'the set of square numbers' with 'the species of numbers picked out/constructed by forming squares of integers'. The latter gives no sense of a pre-determined totality.

2. See discussion on pp. 18 & 22 above and footnote 2 on p. 45 below.

And Brouwer claims that 'neither the ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary...'. The reason for this, according to Heyting, is 'the fundamental ambiguousness of language. As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts. '...Thus the mathematical language is always to be mistrusted, and the only thing that really counts in mathematics is the inner state, the mental mathematical construction.' (Klenk, p. 19, with quotations from Brouwer and Heyting).

The strength of Brouwer's programme rests on the acceptance that 'the mental mathematical construction' requires no formal justification but is a self-validating procedure. Once accepted, natural numbers and fractions arise as mental constructs without the intrusion of antinomies. Logicists may criticise this method of mental constructibility as unbearably complex, but Brouwer claimed that this was due to their lack of familiarity with the system. This movement has its attraction in not attempting to provide a complete answer to 'mathematics', but to accept that man will be left at any time with incomplete knowledge of propositions. Language will be always an imperfect vehicle for communicating results constructed in the mind.

One further principle of Brouwer's programme that requires exposition before one considers the two questions taken as central to each of the first four chapters,[1] is the rejection of the Law of Excluded Middle as being of unrestricted application. In other words, Brouwer claims that one may not be able to construct a proof of a proposition p's truth nor

1. The two questions are, 'Are there eternal objects of mathematics?' and 'Is "mathematics" logically distinct from the "empirical sciences"?'.

construct a proof of the impossibility of p's truth. There is a distinction to be remembered between the semantic principle: 'every statement is either true or false', which Brouwer rejects, and the logical law 'p v - p' which Brouwer accepts. Thus Brouwer is rejecting the principle that in mathematics every problem is solvable.[1] He argued that all previous mathematicians had taken this to be a working assumption and in so doing, were Platonists by default, at least. That is, they were assuming that in some eternity all of mathematics already existed, waiting to be uncovered. This realisation alone showed the quality of Brouwer's insight.

QUESTION ONE: Are there eternal Objects of Mathematics?

All intuitionists including Brouwer accept the idealism that Kant characterised as being the ontological foundation of mathematics. For Kant, the mathematical entities are concepts constructed *a priori* in intuition, and this notion of constructible entities of the mind is central to all intuitionism.[2] Thus the objects of mathematics are mentally constructed series of abstract entities, produced in the on-going expansion of the system as constructed at any instant. The totality of mathematics is always in 'flux'. Classical mathematics is the passive uncovering of *a priori* or empirically existent sets while intuitionist mathematics is

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1. It is this position of complete decidability that Hilbert was so strongly committed to, and which Goedel proved was a hopeless dream. That is, there are mathematical problems which are not solvable in an absolute sense. Heyting argues that accepting some proposition of mathematics as insoluble, is to reject the Law of Excluded Middle, for there is then a 'p' which is neither true nor false (see Heyting's *Intuitionism*, p. 2).

2. Kant's position on this point was outlined in pp. 18 to 20 above. Thus 'mathematical knowledge' describes 'space and time' which are particular universal categories of human experience. These are 'particular' for they are divisible as an apple or an orange is divisible. Similarly, their parts are describable mathematically. A part of space is described by three co-ordinates say, and a part of an apple by a fraction. Each part has a mathematical description. (See Kant's elaborate discussion of this in *Critique of Pure Reason*, Method of Transcendentalism, Chapter 1).

the active generation of more species.[1]

The clearly critical problem for Intuitionism is the prevention of a solipsistic reduction. Entities constructed in the mind are not open to public view and so one seems to be discussing a number of privately produced mathematical words whose status is open to the attacks of Wittgenstein's 'private language argument'.[2] The only basis of objectivity for the Intuitionist is the belief that everybody is capable of having the same self-evident experience. It is pleasing but not part of mathematics when reports of such mental activity, by several different people, coincide.

Dummett provides a lucid picture of the alternative to transcendental and empirical realities of mathematics, 'If we think that mathematical results are in some sense imposed on us from without, we could have instead the picture of a mathematical reality not already in existence but as it were coming into being as we probe' (see Dummett, p. 18). This contrasts the passive view of mathematics with the active, which is supported both by Intuitionists and by Wittgenstein. The difference of interpretation between Intuitionists and Wittgenstein is that the entities are constructed in the mind for the former, and in a public language for the latter. Wittgenstein may be seen as rejecting

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1. A contrast is easily drawn between a 'creative approach' in mathematics and a 'discovery approach', and this will be developed in part 2 of the thesis. The point indicated here is the susceptibility of Intuitionism to a description of 'mathematical constructibility' as 'creating' or possibly, 'inventing'.

2. In the Philosophical Investigations, pp. 53ff. one finds Wittgenstein's specific rejection of total reliance upon 'mental knowledge', and he employs there examples from mathematics. The point that Wittgenstein drives home, is that I cannot be sure that what I intuitively claim to know is not a figment of my imagination, without public confirmation. Heyting would agree, that there is no guarantee that what is publicly confirmed is an accurate representation of the mental construction, but the mental construction is essential to all Intuitionists. Wittgenstein argues that it must be inessential, for objectivity to be preserved (See Remarks on the Foundations of Mathematics, V, 6). This brings the argument up to the question of 'truth' which will be tackled in Chapter 5 below.

the philosophy of mathematics presented by the Intuitionists, while accepting the reports of the mental constructions as being mathematics itself. This might be called 'Intuitionist Formalism'[1] (see Koerner, p. 139) - 'Formalism with a Human Face'. To agree that there are no 'real' mathematical entities may matter far less for the education of children than agreement or disagreement as to the purpose of mathematics itself. It is this point that will be further resolved, as one considers the Intuitionist response to the second question.

QUESTION TWO: Is 'mathematics' logically distinct from the 'empirical sciences'?

Brouwer's response to this question was indicated by the comment on p. 43 above, that 'mathematics in itself is autonomous'. Brouwer indicates a logically necessary connection between mathematics and the category of 'time', but no such connection with either empirical sciences or logic. Brouwer is not thereby prescribing an answer to the question which he would consider far less important, though a natural consequence of Intuitionism, 'Is "the reporting of mathematics" logically distinct from the "empirical sciences"?' This chapter will consider answers to both[2]

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1. It might be more fully presented as 'normative Intuitionist Formalism', for Wittgenstein denies that mathematics consists either of 'real entities' or of 'ideal entities', rather it is essentially a language in use. That this is not a reduction to Logicism or Formalism is made clear by Wittgenstein's warning that one is not discussing 'facts' in mathematics but the use of rules. Thus the law of excluded middle is just one more rule which one chooses to include, but one may choose also, as a community to add further rules to limit its use. Nothing is pre-existent in mathematics, not even a logical principle like the law of excluded middle (see RFM, IV). It has been argued above, p.31, footnote 2, that neither is this to equate Wittgenstein's position with that of the Strict Formalists, for he is intensely concerned to preserve the human element - a proof holds only if men are convinced by it and not simply if and when a computer produces it. Nor is Wittgenstein's position a form of Russell's nominalism, for Wittgenstein's overall commitment is to the consideration of propositions rather than to abstractions from reality (see footnote 1, p. 18. Compare Anscombe's interpretation of Aristotle, with Wittgenstein's position in RFM, IV, particularly pp. 139-43).

2. These questions are distinct for one refers to the status of mathematics itself and the other to the status of the reporting of mathematics. The one may be independent of empirical sciences without both being so.

questions and explain how someone heavily in Brouwer's intellectual debt, like Wittgenstein reunites the two questions.

Brouwer's most immediate follower, Heyting argues that 'for every logical theorem: it is but a mathematical theorem of extreme generality; that is to say, logic is a part of mathematics, and can by no means serve as a foundation for it' (Heyting, p. 6). The logic of mathematics arises as inductive generalisations about those mental, mathematical constructions already achieved. This logic is called 'intuitionistic logic'. Commutativity is first demonstrated for integers (2 + 3 = 3 + 2, 4 + 7 = 7 + 4...) and then for real numbers say, and so on. Heyting is explaining that in the same way as there are no pre-existent mathematical entities, there are no pre-existent laws of intuitionistic logic, for constructing entities. This is to be contrasted with the reporting of the mental constructions which 'express purely empirical results...a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. "2 + 2 = 3 + 1" must be read as an abbreviation for the statement: "I have effected the mental constructions indicated by "2 + 2" and by "3 + 1" and I have found that they lead to the same result.' (Heyting, p. 8). This report is logically equivalent to the report of an experiment in any science. The logic of ordinary language underlines any reporting, including mathematical. Such reporting can only be contingently related to whatever is reported. Thus reporting mathematical events and reporting scientific discoveries are not logically distinct kinds of reporting.
To summarise Heyting's responses to the two forms of the second question,

a) Mathematics is logically distinct from the empirical sciences and from logic (i.e. the logic underlying everyday discourse - as against 'thought').

b) Mathematics reporting is logically coincident with the reporting of all empirical sciences and is itself an empirical activity. It amounts to 'a scientific examination of intuitionistic mathematics', a description of mathematics but not mathematics itself. This description will obey the logic underlying everyday discourse, for it is given in an extension of such a discourse.

Heyting asks the question in his book, Intuitionism, what is the point of 'intuitionism' in itself. The reports may be employable in the real world but the autonomous mental mathematical constructions are isolated from the world and as such, can hardly be useful to it. Heyting responds that Intuitionism in itself may not be valuable to the empirical sciences but yet may facilitate the developments in 'philosophy, history and the social sciences' for 'mathematics...is a study of certain functions of the human mind' (Heyting, p. 10). Heyting is wishing to stress overall the intrinsic value of intuitionistic mathematics, as one might stress that of art. This has clear implications for the attitude with which one may teach mathematics to others, and this will be a major concern in part 2 of the thesis.

On the other hand, if one treats the mental constructions as inessential rather than intrinsically valuable, then a strikingly different conception of mathematical value will arise. Such a position is that held by Wittgenstein. Wittgenstein seems to return to Kant of whom the Intuitionists lose sight, when they discuss the value of mathematics. Kant had argued that mathematics is logically necessary for 'pure natural science' (Newtonian mechanics in particular), which provides the only correct description of the world. Wittgenstein is not supporting the uniqueness claim, but is arguing that mathematics provides the forms in which the world is describable.[1] This is the paramount reason for valuing

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1. Koerner gives Weyl as an alternative to the Intuitionists' view of the value of mathematics (Koerner, pp. 144ff.) and that Weyl attempts a modification of Kant's position in the light of nearly two hundred years of scientific advancement. Weyl was initially an Intuitionist who came to have increasing sympathy with modifications of Formalism (see Hermann Weyl, Philosophy of Mathematics and Natural Science, Princeton, 1949).

mathematics, as it opens up for man dimensions to life that would be non-existent otherwise. For example, the British would have a far less rich 'language-game' of discussing the weather if isobars, thermometers, etc. could not be measured. As Klenk concludes in her support of this position of Wittgenstein's, 'Mathematics...provides us with a conceptual framework into which we can fit our empirical experience' (Klenk, pp. 66-70). From a teaching point of view, Wittgenstein leaves one in no doubt that mathematics can be a vital activity which enables the enrichment of one's empirical experiences. He is stimulating a concern for 'using mathematics' in stark contrast to the isolationism of Brouwer.

However, Wittgenstein accepts the Intuitionist view that mathematics is synthetic _a priori_,[1] and as such is logically distinct from both the empirical sciences and the logic underlying all discourse. While Heyting and Brouwer distinguish 'mathematics' and 'reporting of mathematics', Wittgenstein considers the mental constructibility to be a possible, but not an essential feature of the total mathematical activity. The written or spoken features of the activity are not a report of an inner experience but the essential feature of mathematics itself. In addition, an essential feature of mathematical proof is human understanding of the written or spoken activity, sufficient for any person actually to be convinced by the proof (this is the key sentence for an explanation of Wittgenstein's theory of truth and will be focused upon in Chapter 5 below. Thus Wittgenstein collapses the notions of 'mathematics' and 'mathematics reporting' as one activity with two senses. Writing out '5 x 3 = 15' is both a calculation and the assertion (or reporting) of a _rule_ for

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1. 'The distribution of primes would be an ideal example of what could be called synthetic _a priori_, for one can say that it is at any rate not discoverable by an analysis of the concept of a prime number' (_RFM_, III, 42).

others to follow.[1] The result for Wittgenstein is that the second question has the one answer, 'Mathematics is logically distinct from the empirical sciences.'

CONCLUSION

It would be reasonable to ask how Wittgenstein has come to play an important part in this chapter when he denies the central condition of Intuitionism, its special identity of 'mental constructions' and mathematics.[2] What is suggested here is that a natural sequence occurs from the work of Kant, through the work of Brouwer to the work of Wittgenstein.[3] Both Brouwer and his followers and Wittgenstein came to accept Kant's belief in the synthetic *a priori* nature of mathematical propositions. They both accept Kant's identification of mathematics as an activity of construction rather than uncovering. Wittgenstein may be seen as developing away from Intuitionism in so far as he argues that 'mental constructions' lead to unanswerable questions of objectivity, unless seen as

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1. The underlining of 'rule' is to emphasise that 'mathematical propositions are instruments taken up into the language once for all...' (*ibid*., II, 29) and are not descriptive of an empirical world. The propositions 'show' a man how to do things, rather than 'say' any truth about things. Educationally it is important to note that Wittgenstein is not designating 'repetition of a rule' as sufficient for one to be said to understand it. One must be able to act 'on the rule' as well as in 'accordance with' it. One must appreciate its relevance, 'But does the proof only bring us to the point of going by this rule (accepting it), or does it also shew us HOW we are to go by it? (*ibid*., II, 28)...every proof, each individual calculation makes new connexions!' (*ibid*., II, 47). To understand a mathematical proposition is, according to Wittgenstein, to appreciate its function in a range of contexts. Wittgenstein is making the important suggestion that every proof adds to knowledge and does not merely confirm what was already known. While this point excites, it also removes the traditional grounds for accepting proofs as compelling, i.e. before Wittgenstein, a proof would be seen generally as saying the same thing as was already known but in a new way. Wittgenstein suggests that a proof shows new things and so confirmation cannot be the basis of compulsion.

2. Wittgenstein goes so far as to say, 'Intuitionism is all bosh - entirely.' (*Lectures on the Foundations of Mathematics*, p. 237)

3. The discussion of Wittgenstein in this chapter is to be viewed as an independent development of, and only partial agreement with the views expressed by V. H. Klenk in her, *Wittgenstein's Philosophy of Mathematics* and M. E. Tiles in her, as yet unpublished, article, 'Self-Reference, Saying and Showing'.

inessential features of mathematical reasoning. He admits the possibility of such mental activity but does not rest all his case upon it. Similarly, he warns mathematicians about the unrestricted use of the law of excluded middle, rather than renouncing it.[1] Wittgenstein tries to give mathematics 'meaning', rather than attempting to establish a new mathematics to stand beside classical mathematics, or even replace it as the Intuitionists desire to do. Ontologically some agreement among Kant, the Intuitionists and Wittgenstein is retained, for they all reject the existence of an independent realm of mathematical entities. This is sufficient unanimity for them all to answer, 'No' to question 1. Similarly, there is sufficient agreement for them all to answer 'Yes' to question 2 (at least, in a form that does not explicitly mention the reporting of mathematics). Educationally, the critical point is that mathematics is presented as 'doing' rather than 'seeing' or 'repeating what already is'. In the language of Intuitionism, no one else can 'do' my mental construction which enables me to know that something is right.[2]

Finally, it is to be noted that the first three chapters have brought forth the development of three philosophies of mathematics and the next chapter will attempt to consider less clearly identified movements in philosophy of mathematics arising in the past hundred years. There has been no attempt made to consider movements based on other grounds than philosophical ones, although politics or an intrinsic love of subject

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1. The importance of cautious treatment of the principle of bivalence was elucidated on p. 45 above.

2. This use of 'right' indicates that the argument will remain incomplete until the discussion of the views of 'truth' presented by Kant, Wittgenstein and the Intuitionists. The use of the term 'synthetic <u>a priori</u>' may be seen as temporarily begging the question, but hopefully one can be forgiven for asking 'suspension of disbelief' until Chapter 5.

may be seen as having produced other movements.[1] Once outlines of the various movements have been completed then the material thus gathered can be focused on the question that has been emerging already, 'what is mathematical truth?'.

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1. The only other movement to be mentioned in the thesis is that of the Bourbakists for they have had an influence on the nature of some Modern Mathematics text books in the last twenty years. Bourbaki is the name taken by a group of practising mathematicians who have attempted a formalisation of all mathematics. These mathematicians have made no attempt to assert a particular philosophical standpoint, but the nature of their project implies an implicit rejection of Intuitionism. They prefer to appeal to 'the common sense of the mathematician'. Their formalisation is based upon the Theory of Sets which is assumed consistent, but it is allowed that any contradiction would result in a modification of this theory rather than in mathematics. For all the formalisation there is a certain sympathy with Intuitionism, as the Bourbaki group believe mathematics provides insight into the way people think and thereby uncover 'the most fundamental faculties of the human mind' (quoted by Koerner, of Heyting, p. 121).

CHAPTER 4

MATHEMATICS AS HYPOTHETICAL

INTRODUCTION

It was noted in Chapter 1 that Aristotle built up the concept of 'deductive science'. In the last hundred years this has been given a further twist, and most contemporary philosophers talk of 'hypothetico-deductive sciences'. The fundamental theme of this chapter is the identification by certain movements of what they claim 'scientists really do', and how mathematics acts as a key feature of scientific advancement. These movements are brought together here in the form of one broad school, just like those identified in the previous two chapters, except that this school is the author's own creation. The central linkage of this new school is to be found in the fundamental connection made between mathematics and its application. In this sense, Wittgenstein has provided a natural entry point to this chapter, for he centred his explanations on 'mathematics in use' and sought to retain the human element in what otherwise could be seen as a 'formalised deductive science'.[1] Wittgenstein highlights the rejection of mathematics as something that can be uncovered or discovered, just as much as the Hypothesisors will be seen to do.[2] Wittgenstein sees new areas of mathematics invented in just the way that one imagines rugby was invented by the changing of some rules and the

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1. Tarski's term for mathematics in 'About some fundamental concepts of metamathematics' (Logic, Semantics Metamathematics) and mentioned by Lakatos, p. 3 (Proofs and Refutations).

2. On pp. 67 and 68 below the ontological standpoints of Ormell and Lakatos are identified, and on pp. 46-49 above, there is a fuller discussion of Wittgenstein's own standpoint and his rejection of Intuitionist idealism.

addition of some other rules and concepts. Wittgenstein identifies mathematics with inventions in a language-game and not with the study of any kind of objects that Logicists and Formalists might present. The Intuitionists reject these alternative ontologies too, but retain a mental ontology that Wittgenstein considers inessential. There is a further link with the previous chapter, for like the Intuitionists the movements presented in this chapter, all accept the incompleteness of mathematical knowledge.

The model for these 'Hypothetical movements' as described thus far might be thought to be present in some form of Empiricism, but all these movements are united in their rejection of Empiricism, particularly as envisaged by J. S. Mill. Within decades of Mill writing a System of Logic (1843), the founder of Pragmatism, the oldest movement discussed in this chapter, was presenting a vehement refutation of Mill's thesis.[1] This founder was C. S. Peirce.

The initial section of this chapter is given over to a discussion of the intellectual framework in the midst of which Peirce developed his view of mathematics, in the last quarter of the nineteenth century, and how his view is reflected in the contemporary view of Ormell. The view of another anti-orthodox thinker, Lakatos, is then outlined and shown to have important features in common with Peirce and Ormell. As Frege and Russell have been seen to agree on a basic programme to reduce mathematics

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1. See for example Peirce's comments on Mill in 'Lessons from the History of Science' as reprinted in Essays in the Philosophy of Science, p. 207 (Bobbs-Merrill, 1957). This adversity between Peirce and Mill may surprise those who wrongly identify 'usefulness' as a common element in Pragmatism and Utilitarianism. Peirce has much more in common with Wittgenstein for both saw 'meaning' in terms of 'use'. They define a concept by defining its role. Pragmatism centres on 'use' not utility.

2. C. P. Ormell has been Director of the Schools Council Sixth Form Mathematics Project from 1969-1979 and in that time has provided a clear philosophy of mathematics and of mathematics education, particularly in his article, 'Mathematics, science of possibility', Int. J. Math. Educ. Sci. Technol. 3, 329-41 (1972).

to logic, and thereby both may be called Logicists, even though they diverge considerably in the way each sets about implementing the programme, Lakatos and Ormell can be linked within one movement although they have identifiable characteristics of their own. They are united in the presentation of mathematics as a hypothetical and a human science, and not simply a 'formalised deductive science', but diverge in the stress they each put on human contexts outside mathematics itself. Ormell sees himself following Wittgenstein in as far as arguing that, 'meaning of any form of words (or symbols) stems ultimately from its use in human contexts' ('The Crisis of Meaning in Mathematics' in Mathematics Teacher (India), 11A, 1976, pp. 23-6). While Ormell is keen to look broadly at the human contexts in which mathematics occurs, Lakatos sticks to the internal activities of mathematicians or moves just outside such boundaries to interaction[1] with theoretical physicists and chemists. Finally responses to the two questions[2] given in Chapter 1 above, are outlined and an overall conclusion is drawn.

PEIRCE and ORMELL

Peirce's view of mathematics is dependent upon extensive interaction with his father who was Professor of Mathematics at Harvard. Peirce rejects the full-blooded idealism of his father for a temperant realism. Peirce was well versed in Kant, and his realism stems from an interpretation of the Critique of Pure Reason (2nd Edition), as suggesting the possibility of the direct experience of noumena. This may be seen as a rejection of Mill's view that there are known limits to what man can know. Peirce views science as the struggle for the truths of science. Although these 'general

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1. Ormell's greater concern for interaction across all human contexts, and not just with those at the traditionally considered boundaries of mathematics will be discussed further in part 2, particularly Chapter 10.

2. The questions: 1) Are there eternal objects of mathematics?
   2) Is 'mathematics' logically distinct from the empirical sciences?

principles are really operative in nature' (quoted in Feibleman, p. 59 from C.P. 5.101*), at any time one can never know whether one's present propositions are conclusively true. Hence the conclusiveness of all scientific proposals remains uncertain. Peirce talks of such propositions remaining 'fallible'.

Consequent upon the scientific revolution that had begun with Darwin, Peirce wished particularly to clarify the epistemological foundations of science, and in so doing he laid the foundations of Pragmatism. Peirce was most concerned with the relation of sciences to logic and to mathematics, because of his own work as a logician and his father's, as a mathematician. Thus Peirce came to ask the fundamental question, 'What is the methodology of science?'. It is the similarity in the consequent answers to the sub-question, 'What is the methodology of mathematics?' that explains why Ormell is joined with Peirce in this chapter. While Peirce says, 'the necessary reasoning of mathematics is performed by means of observation and experiment' (Feibleman, p. 138, quoted from C.P. 3.560), Ormell says 'One might describe mathematics as the investigation of the properties of possibilities by means of the manipulation of zq-apparatus.'[1] ('Mathematics, Science of Possibility, p. 339).

Peirce indicates that the Pragmatists accept as a key definition for their philosophy that 'belief' is 'that upon which a man is prepared to act. From this definition pragmatism is scarce more than a corollary.' This is the view of belief originated by the Scottish philosopher,

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* All references 'C.P.' refer to Hartshorne and Weiss, Collected Papers of Charles Sanders Peirce - thus C.P. 5.101 is 'Volume 5 para. 101'.

1. By 'zero-quasi apparatus' Ormell is reminding one implicitly of Wittgenstein's comment that 'In mathematics process and result are equivalent' (RFM, I, 82). That is, the apparatus manipulated is also 'the end of the process' itself. Ormell would seem to be reinforcing the distinction that it was noted Aristotle draws, between mathematics and science, 'in actuality' (See p. 21 above). Ormell is identifying the fact that the apparatus of mathematics has no real existence (except 'zero-quasi' existence), unlike the test tubes and glass prisms of the natural sciences.

Alexander Bain, to whom Mill refers with great respect in his autobiography. This concern to see things achieved was common to Bain, Mill and the Pragmatists, including Peirce. However Peirce was no anti-intellectualist, for he argued that knowledge is off-course if its 'purpose is not the solution of great problems, but merely the fitting of a selection of young men to earn more money than their fellow citizens...' and to emphasise this point one may quote a consequent remark that, 'True science is distinctively the study of useless things' (Essays, p. 210).[1]

Peirce was not satisfied with the uncovering of a methodology but was asking the further question, 'what is the purpose of this branch of knowledge?'. It is this question that draws Peirce close to Wittgenstein, rather than to the Logicists and Formalists discussed in Chapter 2. The relationship may be clarified if one first considers further Peirce's realist position. As a realist Peirce has answers to both ontological and epistemological questions and he moves from 'general solutions' to specific ones about mathematics. A similar route will be taken here.

On p. 57, it was noted that Peirce asserted that 'general principles are really operative in nature' and are not simply imposed by man on nature. Peirce does not claim originality for this view but says, 'That is the view of scholastic realism.' In rejecting what he saw as Mill's nominalism, Peirce saw himself as reiterating Duns Scotus' defence of realism against the attacks of Occam's nominalism. In contemporary terms,[2]

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1. These points indicate features of the discussion of intrinsic/extrinsic values to be found below, in part 3 of this thesis. Clearly 'fitting men to earn money' is extrinsic, and 'studying something useless' is likely to indicate intrinsic value, but of particular interest, in view of later discussion, is the kind of value placed on studies done for 'problem-solving potential'.

2. 'The problem of universal laws and their truth; that is, the problem of regularities' (Popper, Unended Quest, p. 19). It was noted in footnote 2, p. 17 and footnote 1, p. 18 above, how similarities may be drawn among Aristotle, Locke and Russell. Each of them has been identified by some critics as 'nominalist'. Each man attempts the presentation of a thesis free of 'platonic entities' but accepts the idea that universals (numbers are abstractions from experience) are

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Peirce's realism is based on the argument that regularity in nature is there in itself, and is not just a psychological need that man imposes. The nominalists however claim that the only kinds of things existing extra-mentally are individuals. Furthermore, they argue, the only meaningful alternative position is that of Platonic realism. Peirce follows the Platonic realists (like Frege) in rejecting psychologism[1] but stops short of asserting the existence of 'forms'.[2] However, later Pragmatists, from William James on, were to accept psychologism. Thus it is no surprise to find Dewey writing, 'the status of mathematics is as empirical as that of metallurgy. Men began with counting and measuring things just as they began with pounding and burning them' (Reconstruction in Philosophy, p. 137), for his pragmatism is at one with Mill's Utilitarianism[3] on this point.

Fn. 2, p. 58 contd.
logical fictions. The alternative argument is that there are meaningful propositions that neither refer to nor are logically detached from experience. This is the position presented most clearly by Wittgenstein and discussed on p. 47 above. Like Wittgenstein, Peirce sees these universal propositions as 'laws' in the sense that (Wittgenstein calls them 'rules') they 'show a man how to do things rather than 'say' anything. Peirce identifies a 'say'/'show' distinction between logic and mathematics. 'The one studies the science of drawing conclusions, the other the science which draws necessary conclusions' (Essays, p. 266). Logic 'says' what mathematics 'shows'. 'Logic' is identified in Kantian terms for Peirce. It is a narrower view than the Fregean perspective taken by Wittgenstein, by which logic covers all human reasoning. This distinction has educational implications. In Kantian terms, being 'logical' entails being able to 'show and say' one's reasons, while Wittgenstein can be satisfied by 'showing' alone. Tighter and looser conceptions of 'understanding' may follow, and this will be returned to in some depth in part 3 of the thesis.

1. 'Psychologism' is taken here as the view that some words necessarily arouse mental images relevant to their meaning. Its rejection is taken as the denial of any necessary connection between rules of meaning and principles about thought.

2. See Haack, S., Philosophy of Logics, p. 55, for a similar interpretation of Peirce, on this point.

3. Dewey is clearly sympathetic to Mill and writes on p. 183 of this book, 'The idea of a fixed and single end lying beyond the diversity of human needs and acts rendered utilitarianism incapable of being an adequate representative of the modern spirit.' In other words, Mill was going in the right direction but stops short of Dewey's view of pragmatism, in which 'The hypothesis that works is the true one...' (Reconstruction in Philosophy, p. 156).

Once one recognises Peirce's belief that there is truth for which a man of science searches, one is not surprised to read, that on the basis of 'few observations of a given matter, and those rough ones, a law is made out which, when the observations come to be increased in number and made more accurate, is found not to hold exactly' (Essays in the Philosophy of Science, p. 219). Peirce stresses 'observations' but not just for the physical sciences, for 'the necessary reasoning of mathematics is performed by observation and experiment' (C.P. 3.560). (Underlinings are not in original texts.) This approach could be equated with Mill's, in which induction is the central method of all science including mathematics, but Peirce admits induction as only one method among many, for 'The best hypothesis, in the sense of the one most recommending itself to the enquirer, is the one which can be the most readily refuted if it is false. This far outweighs the trifling merit of being likely...which falls in with our preconceived ideas' and may involve 'strategems for cutting off inquiry' (Essays, p. 22). Peirce gives a broad description of scientific method including the pro's and con's of verifiability and fallibilism.[1] The picture of science that results is that of an activity with built-in self-correction by which means it is directed to the truth, but could never know it has reached it.[2]

It is with this immense concern for 'testability' that Peirce considers mathematics. He sees it differing strikingly from all other sciences because it is, 'Cut off from all inquiry into existential

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1. While attacking Comte and Poincaré as strict verificationists, Peirce expects any scientific result to 'be of such a nature that it could not occur without being detected' (Essays, p. 255). In Peirce's works one finds indications of positivism but there are stronger signs of the ideas of Popper and Lakatos, except in their rejection of induction. Peirce seems to follow a middle course between strict verificationism and falsificationism.

2. Compare Popper who writes, 'Explanation is always incomplete...the new why-question may lead to a new theory which not only 'explains' the old theory, but corrects it. This is why the evolution of physics is likely to be an endless process of correction and better approximation...' (Unended Quest, pp. 130-1).

reality...mathematics is only busied about <u>purely hypothetical questions</u>' (Feibleman, p. 353 quoting from <u>C.P</u>. 1.53). All sciences including logic, are to be recognised as having a mathematical branch. This reiterates the distinction between Peirce's Kantian view of logic and Wittgenstein's Fregean view. Peirce views logic as one science among many while mathematics is the 'showing' feature of all sciences. Wittgenstein presents logic as all-embracing for human reasoning and mathematics is depicted as only one corner of this.

Peirce sees the sciences as locating the facts of sensible experience in the world of ideas while mathematics consists of 'mental experiments', 'ideal experimentation' and 'abstractive observation'[1] in an 'arbitrarily hypothetical universe', and could be seen as 'an advanced theory of graphs which...treat(s) of the whole universe of logical possibility'. (See Feibleman, pp. 138-40). To summarise, 'Mathematics is the study of what is true of hypothetical states of things. That is its essence and definition.' (<u>C.P</u>. 4.233).

It is to be hoped that Peirce's view of mathematics as hypothetical has been drawn sufficiently clearly for connections to be brought out now, between Peirce's work and the contemporary views of Ormell. <u>Ormell</u>[2] is writing some fifty years after Peirce's death in 1914, and after Logicism, Formalism and Intuitionism have grown up and presented programmes. In addition, Peirce's programme Pragmatism, has taken on new directions and there have been numerous reactions to these and all the other programmes. The result is that one may be surprised to find any contemporary views remaining independent of the main movements covered in Chapters 2 and 3 above.

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1. By 'abstractive observation', Peirce might be seen as accepting 'nominalism'. However, what is abstracted is not the name of something, but the law by which constructions occur.

2. For a detailed analysis of Ormell's position, see my article, 'Mathematics, Science of Possibility - A Critical Review', <u>Int. J. Math. Educ. Sci. Technol</u>., 4, 413-20 (1973).

Ormell follows Peirce in being particularly concerned to identify mathematics as the practising mathematician sees it, rather than as philosophers view it. Ormell accepts that one has the option of taking mathematics as a nominalist system - the manipulation of symbols without references. He sees this position as compatible with Formalism[1] as discussed in Chapter 2 above. However, Ormell prefers to talk about 'interpreted mathematics' rather as Hilbert had sought to axiomatise only those formal systems that had some mathematical interpretation. Ormell sees the Formalistic metamathematicians of Chapter 2, as blind to half of what mathematics is to most mathematicians, and certainly what it <u>ought</u> to be viewed as, by most ordinary citizens.

For Ormell the objects of mathematics are 'kits' packed with undiscovered potential. Ormell talks in terms of 'potential model expressions' which may be applied either to another branch of mathematics (algebra to geometry for example) or to an outside area (as in designing a bridge, say). Thus Ormell sees the key stimulus for mathematical development in the question, 'what are the possible applications of a given model?', rather than 'what new formulae does this formal system allow one to generate?', as the strict Formalists would be expected to see things. Like Peirce, Ormell centres his view of mathematics on its hypothetical nature, believing that this allows any element of mathematics to have a potential usefulness, as part of 'the science of possibility'.

<u>LAKATOS and THE HYPOTHETICAL SCHOOL</u>. <u>Lakatos</u> provides an equally contemporary, but much more strikingly independent view of mathematics than Ormell. In the same way that Popper remained outside the most readily accepted circle of Philosophy of Science, as it came from Vienna to England and to America, so in the same way Lakatos remained aloof from

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1. A specific discussion of Formalism and this standpoint is found in my article, 'Mathematics, Science of Possibility - A Critical Review', pp. 414-16.

any kind of formalism, standing out for 'informalism'.[1] Lakatos contrasts 'the monotonous increase of truth' that occurs in a mechanically decidable formal system, with the enlightenment that occurs when one finds that a connection holds for a more limited domain than one previously believed. Imagine the spectrum that opens for the person who discovers that 'p/q' (where p and q are integers) is not sufficient to describe all numbers, for there are irrationals like $\sqrt{2}$. Getting things right is seen by Lakatos as a poor description of a mathematician's task.

One sense in which Lakatos may be identified as an Hypothesisor follows naturally from his abhorrence of an over-emphasis upon 'getting things right'. His view is not one of absolute certainty, but of truth accepted relative to a logic which is at present admitted. This is the essential nature of deductive proof based upon the hypothesis that this logic will do. 'Certainty is never achieved'[2] (Proofs and Refutations, p. 56). Another form of proof is that in which a proposition has at present no counter-example and none is foreseeable. The hypothesis in this latter case is that no counter-examples will occur. The first form of hypothesis is exemplified in proving Pythagoras' theorem for Euclidean geometry, and the latter form is exemplified in accepting 'all bachelors are unmarried'.[3] Lakatos puts 'problems' at the centre of informal mathematics. Mathematicians invest systems or parts of systems in order to deal with a problem.

Having given brief outlines of the views of Peirce, Ormell and Lakatos, certain features can be picked out as common to all three.

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1. 'An investigation of informal mathematics will yield a rich situational logic for working mathematicians, a situational logic which is neither mechanical nor irrational, but which cannot be recognised and still less, stimulated, by the formalist philosophy.' (Proofs and Refutations, p. 4).

2. Popper and Peirce agree, see p. 60 above, that certainty is never known in science, and Lakatos extends this lack of certainty to mathematics.

3. That this is an acceptable example for Lakatos is indicative of his rejection of the analytic/synthetic distinction, about which comment is made in the next chapter.

'Hypothesising' is seen as the key activity for mathematics, rather than 'concrete experimentation' as is found in other sciences. Hypothesising indicates the illuminative power that Lakatos and Ormell see in mathematics. Mathematics is developed by the production of hypothetical structures. Ormell sees them as acts of 'disciplined imagination', and Lakatos calls them 'thought-experiments'; and they provide clarity where it is lacking and can even make advancements where physical experimentation is practically impossible. By hypothesising, one gets to know more about continuity in algebraic functions,[1] or the feasibility of inflatable bumps for improving road safety.[2] If someone fears a difficulty in their work or has a practical problem to resolve, through the use of some part of mathematics in fresh pastures, then there is the possibility of further illumination and the development of mathematics. Thus, it can also be said that all three thinkers, Peirce, Lakatos and Ormell, tie mathematics closely to the purpose of solving problems.[3] 'Hypothesising' is indicative of the ontological framework that unifies this school. A universe of mathematical entities is identified which consists neither of 'solid' objects nor of 'mental' ones. Popper presents such a world and calls it 'World 3'.

<u>World 3</u> is neither an 'idealist' nor a 'Platonic' reality but 'the world of statements in themselves', distinguished from 'thoughts in the

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1. The 'principle of continuity' comes in for lengthy discussion in <u>Proofs and Refutations</u> (pp. 127-131), to exemplify the method of proofs and refutations. By the hypothesising instrument of numerical analysis, much 'graph-commonsense' was disturbed.

2. <u>Teaching Mathematics Applicable, Introductory Guide</u> (p. 21). This example is most interesting because it has been discussed seriously, even in parliament, since (although not because of) 1974 when it was put in a specimen examination paper.

3. Ormell admittedly holds some value in the 'game-side' of mathematics, and so is willing to accept that mathematics can also advance through formal derivation, followed by hypothesising. Yet formal development cannot occur totally without direction, for mathematics is not comparable to absent-minded or spontaneous doodling. Lakatos concurs on this latter point when he rejects the idea that mathematics advances through either 'the rationalism of a machine' or 'the irrationalism of blind guessing' (<u>Proofs and Refutations</u>, p. 4).

sense of thought processes' (which seems to be the place of Intuitionist mathematics). Popper distinguishes three worlds in his book, Unended Quest (pp. 183ff.), and uses the example of a book to clarify his position. In World 1, one finds the physical object that is called a book. In World 2, the thoughts that are stimulated by the words in the book, and in World 3, one finds the content of the book itself.[1]

Both Ormell and Lakatos give clear signs that they would fit mathematical entities into this world 3. Popper describes this world as 'the world of all that man may think up', and this seems to fit particularly well with Ormell's hypothesising imagery as characterised in the phrase 'disciplined imagination'. Through 'disciplined imagination' a mathematician creates models that await application, but they are not simply figments of the imagination for they are open to public scrutiny. The propositions of mathematics represent a world of 'possibilities' which Ormell can surely only claim to be 'reifiable', as he does in his article, 'Mathematics, Science of Possibility', in just that sense identified by world 3.

Lakatos has similar concern for the publicity of the 'thought experiments', which he sees as inventions stimulated to solve given problems. He has no intention of falling foul of the privacy problems of Intuitionism (Lakatos' argument on p. 52 of Proofs and Refutations makes this clear). Lakatos' position may seem at first sight to be asserting ideal, mental constructs, but like Wittgenstein and more particularly like Popper, he unifies what one thinks with the language one speaks. Mathematical proofs do consist of 'thought experiments', but their public language-form must be analysed for counter-examples, 'refutations', before it makes sense to talk of there being a new

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1. Peirce distinguishes the 'universe of logical possibility...the ideal world' from the 'real world' (C.P. 3.527), just as Popper distinguishes World 3 from World 1. Quinton draws parallels between Popper and Peirce along similar lines, in his lecture printed in R. S. Peters (ed.) John Dewey Reconsidered, p. 2.

mathematical proof. It is by this unifying of 'proof' with 'refutation' that Lakatos clearly separates himself from the world 2 of Intuitionism, and stands more comfortably in world 3, with Ormell.

The Purpose of Mathematics. Common to Peirce, Ormell and Lakatos is a belief that mathematics has a potential usefulness, at least. This usefulness can apply either from one part of mathematics to another or from mathematics to the outside world. However Lakatos is to be contrasted with Ormell in particular, on the generating of mathematics. Lakatos seems to follow a line found in Popper[1] that mathematics develops in order to solve problems, while Ormell argues that the mathematics developed has the potential to solve problems. In this sense, the gap between Lakatos and the Logicists and Formalists presented in Chapter 2, is far greater than that between Ormell and Logicists and Formalists. Ormell does not reject the possibility of generating new mathematics[2] without any problem being in sight. Lakatos argues that there are always steps, even in mathematics, which are chosen because of the problem that one is considering, even if the problem is not consciously visible.

Given that one now has a picture of a broadly linked school of thought centred on the hypothetical nature of mathematics, and its concern with the solution of problems, it is possible to consider the two questions identified in Chapter 1.

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1. Popper's falsificationism leads him to write, 'Explanation is always incomplete...the new why-question may lead to a new theory which not only "explains" the old theory, but corrects it.' (Unended Quest, p. 130). On p. 60 above, it was noted that this leads Popper to a perpetual view of scientific evolution, and it would seem this model fits for Lakatos as perpetual mathematical evolution, through 'proof-analysis'.

2. Ormell and Lakatos will be seen closer on this point when the nature of proof is discussed in the next chapter. For both of them a proof cannot be adequate unless it has a demonstrable application, very much in line with the Pragmatist linking 'proof' to 'successful action' (Essays, pp. 252-4).

QUESTION ONE: Are there eternal objects of mathematics?

One might suggest that the ontology for mathematics common to Peirce, Ormell and Lakatos is one that identifies mathematical entities as 'real possibilities'. This phrase picks out the point that the entities are both 'hypothetical' and 'publicly identifiable'. While these entities depend upon man's drawing up hypothetical states of affairs for their existence, the mathematician has not complete control of them. As Peirce says, 'His hypotheses are creatures of his own imagination, but he discovers in them relations which surprise him sometimes' (C.P. 5.567). To answer the question, one is certainly directed to a denial of Platonic realism for mathematical entities have neither eternal nor matter of fact existence, but only the existence provided by man's 'imagination'.[1]

Ormell admits in his article, 'Mathematics, Science of Possibility' that he has had a difficult struggle to find an alternative to Platonic reality, but believes that the 'possibilities' which are the constructs of 'disciplined imagination' are distinguishable from eternal forms, precisely because no such eternal claims are being made. However, Ormell accepts that nominalism is a valid interpretation of the ontology of mathematics, along the lines presented by Wittgenstein (and discussed in Chapter 3 above). That is, one could take the view that there are no objects of mathematics, for mathematics is not a system of objects, but a system of rules employable in the empirical world. Thus Ormell rejects the idea that mathematical objects are eternal, but rather sees them as 'potential model expressions' existing in 'the world of all that man may think up'.

Lakatos' view of this ontological question is very much in line with those of Peirce and Ormell, in so far as the only possible kind of universe in which they could exist, would be one describable as 'World 3'. Although

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1. Peirce stresses 'imagination's' place in proof on pp. 252-54 (ibid.).

Lakatos accepts that man makes mathematical proofs, and also may discover through them 'relations which surprise him sometimes' (Peirce, as quoted on p. 67 above), for Lakatos these proofs and any other mathematical entities have no long-term guarantee of survival. Lakatos argues that one must 'give up the idea that our deductive, inferential intuition is infallible' (<u>Proofs and Refutations</u>, p. 138[1]). In other words, Lakatos is arguing that any element of mathematics could be removed through some subsequent reorganisation of what is called mathematics. Thus, in terms of the specific question, 'Are there eternal objects of mathematics?' Ormell and Peirce argue that there are no objects existing in a Platonic universe, but if man invents 'equilateral triangles' say, then demonstrated truths[2] about them are guaranteed eternally. Lakatos takes the extreme position that even if today 'our' mathematical universe includes what are called 'equilateral triangles', then this gives no guarantees for the future.

For all their differences, Peirce, Ormell and Lakatos agree that the ontology of mathematics lies between the totally theoretical and the actual empirical worlds, with clear paths to both. Educationally, one

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1. The paragraph in which this occurs has the following editors' footnote added, 'This passage seems to us mistaken and we have no doubt that Lakatos, who came to have the highest regard for formal deductive logic, would himself have changed it. First order logic has arrived at a characterisation of the validity of an inference which (relative to a characterisation of the "logical" terms of a language) does make valid inference essentially infallible. Thus one need make only the first of the two admissions mentioned by Lakatos. By a sufficiently good "proof analysis" all the doubt can be thrown onto the axioms (or antecedents of the theorem) leaving none on the proof itself. The method of proofs and refutations is by no means invalidated (as is suggested in the text) by refusing to make the second of these admissions: indeed it may be by this method that proofs are improved so that all the assumptions that have to be made in order that the proof be valid, are made explicit.' (<u>ibid</u>., footnote 4, p. 138). Essentially the two admissions are that there are no infallible propositions and secondly, that there are no infallible proofs - the editors accept the first only.

2. A full discussion of this point can only occur once the views on the analytic/synthetic distinction and that on the overall conception of truth have been clarified, and this will occur in the next chapter.

is shown signs of mathematics as both intellectually rigorous, and relevant to man's inter-action with the empirical world. In this 'science of possibility', no possibilities are cut-off.

QUESTION TWO: Is 'mathematics' logically distinct from the 'empirical sciences'?

Given that much stress has been put on the links inherent in the Hypothetical movements with the 'empirical sciences', one may well be surprised to find a positive answer to this second question. With these movements an appreciation of the ontology of science itself may be necessary, for an accurate reply to be stated. In this way Peirce's broader philosophy was discussed and not just his philosophy of mathematics. Thus one finds that 'problem-solving' is a thread running through the methodology of mathematics as well as that of the natural and social sciences. Lakatos and Ormell concur in their acceptance of this view of Peirce's.

What is given to Peirce directly, and to others implicitly, by Kant is the view that mathematics involves both 'invention' and 'discovery'. The literary analogy would be that someone 'creates' a spy thriller and critics 'discover' a political analysis within it. 'Creating' and 'inventing' may occur in science, but there are no 'given' entities in mathematics. In this way, mathematics is identifiably separated from the empirical sciences. Yet the invention in either sphere may have consequences recognised only through later 'discovery', as is found in the technological advancements of nuclear energy that provide peaceful applications, or with the side-effects of a drug, like Thalidomide.

In the works of Peirce one finds a more complex view of empirical sciences than that assumed by followers of most other movements. The Logicists, Formalists and Intuitionists seem content with a passive view of science, fully identified in 'verification' and 'classification'. Lakatos' criticism of these movements in the area of mathematics is equally appropriate to science. These movements describe methods of proof rather than the way scientists or mathematicians work. The preliminary informal activities are ignored as logically irrelevant to the logical strength of the resulting proofs. Lakatos' argument is that the consequence of ignoring such informal procedures must be incomplete proofs.[1] The foundations are never more than 'conjectures' - 'hypothetical', but the 'proofs' themselves are empty without them.

In the completion of this argument Lakatos certainly parts company with Peirce and Ormell. Lakatos' argument comes to a biting conclusion in a footnote on p. 143 (the argument has begun on p. 138), he writes,

> Now while Popper showed that those who claim that induction is <u>the logic</u> of scientific discovery are wrong, these essays intend to show that those who claim that deduction is <u>the logic</u> of mathematical discovery are wrong. While Popper criticised inductivist style, these essays try to criticise deductivist style. (Underlining not in the text)

As was noted on pp. 58 and 59 above, Peirce's criticism of inductivism is much more limited than Popper's, but Peirce and Popper agree on the essentially infallibilist status of mathematics, and that there is no one logic of scientific discovery. They disagree however on the total eradication of 'induction'. Lakatos accepts Popper's view of scientific discovery but argues similarly in mathematics.

Lakatos holds an 'heuristic' view of both scientific and mathematical discovery; that is the means and methods of problem-solving as related to scientific and mathematical problems. Lakatos says, 'both are characterised by conjectures, proofs, and refutations' (<u>Proofs and Refutations</u>, p. 74). Lakatos' point is that there is nothing identifiable as <u>the logic</u> underlying the conjectures, proofs and refutations of mathematics, any more than Popper believes there to be of scientific explanations, for

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1. A contrasting but not unsympathetic picture is that drawn by the American philosopher, Israel Scheffler through his notion of programmatic definitions. Thereby, he provides a framework for 'science with doubt' (see his book, <u>the Language of Education</u>, Chapter 1). Lakatos is arguing that programmatic definitions occur even in mathematics, but one only knows the definitions one requires, because of informal mathematics.

ultimately 'guessing' has a central place and there is no uniquely identifiable logic of 'guessing'.[1] 'Guessing' precedes 'thought experiments' in Lakatos' view, but Ormell seems either to conflate the two activities in the phrase 'disciplined imagination', or more likely, to reject the notion of mathematical activity free of obedience to formalised manipulations, as is suggested by the word, 'guessing'. Ormell may put problem-solving at the centre of mathematics as Lakatos does, but Ormell refuses to accept that the underlying deductive procedures of mathematics could be fallible; that is, beyond the fallibility of axioms or a given theory.[2]

In the works of Peirce, Popper and Ormell one finds support for the view that mathematics is separable from the empirical sciences, because it alone has the possibility of infallible proofs. Lakatos, in his actual writings, holds the stronger position that even mathematical proofs are not guaranteed infallibility. The difference may be explained

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1. A fascinating suggestion would be that an attempt could be made to link Locke's work on the logic of scientific discovery, as found in Book IV of _an Essay concerning Human Understanding_ with the contemporary views of Popper and Lakatos. See for example, Locke's distinction between the logic of proofs (syllogism) and that of 'intuitive knowledge', Book IV, Chapter 17. As early as Locke, there seems to have been the suggestion that 'human reasoning' has no single identifiable system of logic underlying it. The views of extreme Rationalists and Empiricists may in one sense be characterised as holding the opposite position to Locke. Their mutual difference lies in the type of logic they identify as _the_ underlying logic, and not whether or not there is one at all. On one side Aristotle had identified 'deductive logic' and Mill, on the other, identified 'inductive logic'.

2. One may doubt the presuppositions but not the procedures for establishing new axioms from old; that is, deductive proof. Underlying this rigidity is the belief that there is one system of logic that describes all human reasoning as Frege, Russell and Wittgenstein believed. This is to be contrasted with the Kantian notion of a transcendental logic covering synthetic _a priori_ propositions, in addition to the logic among judgements which underlies analytic _a priori_ and synthetic _a posteriori_ propositions. In a similar vein, the Intuitionists have been seen to hold to more than one logic too. Intuitionist logic is the logic of thought-experiments generated independently of the logic underlying public discourse. What Lakatos' editors seem to fear is any position that claims 'extra' logics, whether one more or on to total relativism, although the unifying or reductionist force only began with Frege's logicism. Such a reductionist view existed before Frege, but only since has it generated the power of self-evidence.

by a model originally employed by Koerner[1] and amended by Ormell, that scientific proofs have a limited repeatability, while mathematical proofs have unlimited repeatability. Koerner sees the distinction as a logical gap, while Lakatos leaves it as possibly one of degree.

Thus largely, the Hypothetical movement shows a close similarity with the methodology proposed for science, but not an identification with the nature of its proof. The movement distinguishes between logic, as uninterpreted symbolic manipulation and mathematics, as necessarily a system with interpretative potential. To reiterate Peirce, (Logic) 'studies the science of drawing conclusions, the other (mathematics) the science which draws necessary conclusions' (Essays, p. 266), and that mathematics is separable from empirical sciences for it 'is only busied about purely hypothetical questions' (Feibleman, p. 353, quoted from C.P. 1.53), while the empirical sciences are concerned with actual questions, solved by actual experiments.[2]

CONCLUSION

From a concise outline of Peirce's, it has been seen that his view of mathematics is of a hypothetical science. Contemporary support for this standpoint is found in the work of Ormell and of Lakatos. Both stress the informal nature of mathematics as a domain of hypothetical reasoning, in contrast to the tightly restrictive views given particularly

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1. Koerner employs the phrases 'infinitely repeatable experiments' and 'infinite repeatability' in outlining his argument that theoretical discourse, at the heart of which is mathematics, is an idealisation of empirical discourse and logically disconnected from it. One reason for this is that infinite repeatability can only occur in theoretical discourse (see Experience and Theory). Ormell amends this in 'Mathematics, Science of Possibility'.

2. As was already noted on p. 57, footnote 1 above, the nature of experimentation in mathematics as against the natural and social sciences may be identified as different, through the type of apparatus employed. Ormell distinguishes the 'zq-apparatus' of mathematics from the concrete materials of other sciences. In mathematics, there is no separation of the apparatus used in the experiment from the record kept of the experiment, for 'there is nothing in addition to the record' ('Mathematics, Science of Possibility', p. 338).

in Chapter 2 above. However, they do not present a picture of unlimited freedom, but show mathematics constrained by the requirements of publicity for demonstrating proofs and refutations, and also by the fundamental linking of mathematics to the solution of problems. By combining these constraints with hypothetical reasoning in which mathematics is generated, one can see marked out a unified movement, distinct from any identified in the preceding two chapters.

CHAPTER 5

MATHEMATICAL TRUTH

INTRODUCTION

The object of this chapter is to investigate the place of truth in the movements discussed in the previous chapters. In order to do this, it is necessary to describe not only certain theories of truth, but also differing views on the analytic/synthetic distinction. This distinction is often critical to theories that recognise more than one kind of true proposition. The most obvious of these distinctions would be one between tautologies and all other true propositions.

Russell suggested that in discussing theories of truth one must bear in mind the distinction between <u>definitions</u> of the concept and <u>criteria</u> for its correct application. One could imagine a child ignorant of the definition of 'thank you', being told that it is what one says whenever an elder gives you something, and not to be said when giving someone else something. All such situations are thereby divided into sets according to a criterion, without knowing a definition. Similarly, it may be hoped that anyone could be given a criterion of truth, so that all propositions[1] may be divided into true or false ones, independently of whether or not one has a definition of truth. The first person traditionally taken to have provided a definition of truth is Aristotle, who wrote, 'To say that what is is not, or that what is not is, is false and to say that what is is, or what is not is not is true;...(<u>Metaphysics</u>,

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1. The use of 'propositions' should not be taken as prejudging any issues about distinctions among 'propositions', 'statements' and 'sentences'. Russell draws the definition/criteria distinction in terms of 'the <u>nature</u> of truth and a <u>criterion</u> of truth' ('On the Nature of Truth' in <u>Philosophical Essays</u>, p. 149).

Book IV, Sect. 7, p. 71 in Bambrough edition). This may be a help if someone makes an assertion like 'the cat is on the mat' and one can ask the question, 'is it?' of oneself, but it does not help one to put assertions together for oneself. It is this latter problem that so concerned Leibniz[1] who imagined the possibility of a computing machine producing such assertions. He soon realised that some propositions could be produced in this way, 'truths of reasoning', but not others, 'truths of fact'. The latter group would involve a non-denumerable number of conditions in the argument, and so no machine could be sure to assert them. In contemporary philosophy one is more used to the distinction highlighted by Kant, between analytic and synthetic propositions, and this is the distinction that is generally used in the thesis.

As the rest of the discussion of mathematical truth[2] will be held up without some clarification of this distinction, this will be tackled briefly first. In the following section, each movement will be taken in turn and it will be argued that certain views of truth come through more strongly in some movements rather than in others. The chapter concludes with the attempt to provide a model of a true mathematical proposition for each movement, in the light of the first part of the thesis as a whole. It is finally indicated that the ontological positions of each movement will emerge as central characteristics of teaching approaches, discussed in the second part of the thesis.

THE ANALYTIC/SYNTHETIC DISTINCTION

Before one identifies differing notions of analyticity, it is important to make clear that the analytic/synthetic relationship is not

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1. One hopes that the reader will recognise the obvious reason for picking out Leibniz, given his connection with Logicism. He identifies 'two kinds of truth' in the Monadology, p. 9 (Everyman Edition).

2, In the Appendix, pp. 273-283 four theories of truth are described. This does not cover all the theories that exist but, hopefully, all those that have a bearing on the mathematical movements identified in the body of the thesis.

synonymous with the a priori/a posteriori one, which concerns the way one comes to know something, rather than whether or not what one claims to know, is true or false.

Thus an a priori proposition requires no empirical investigations for one to know which truth value holds for it. It only concerns terms within languages (taking 'languages' as broadly as Wittgenstein could possibly have done). The paradigm example is 'a bachelor is an unmarried man', which one can construct correctly if one has a dictionary and some rules for English language construction. This is not the case with 'Essex beat Surrey last Saturday'. This a posteriori proposition requires information beyond that attached to language construction, for it to be produced with meaning. However, being told that the first proposition holds a priori, and the second a posteriori, may not be sufficient information for one to know how to set about finding out whether or not either is true. One must further consider whether or not being told that the first is analytic and the second is synthetic will enable one to do this.

Four views of necessary truths will be seen to ground the discussion of mathematical truth in the main body of this chapter. These four views may be summarised as,

(1) A truth of reason is true because its denial would be self-contradictory, and this is either self-evident or the proposition can be reformulated by identities to be self-evident (Leibniz).

(2) An analytic proposition is true because the concept of the predicate is included in the concept of the subject. These are not the only necessary propositions, for the principles of the pre-conditions of all human experience are synthetic a priori propositions (Kant).

(3) An analytic proposition is either a logical truth or is deducible from logical truths (Frege).

(4) An <u>analytic</u> proposition is not a clearly identifiable notion and even if the purported distinction were clarified it would remain empty (Quine).

VIEWS OF MATHEMATICAL TRUTH

In this chapter links will be considered between views of analyticity and the four mathematical movements described in chapters 2 to 4 in order to clarify the differing view of truth, to which they most readily relate. In chapters 2 to 4, it was made clear that certain ontological positions were compatible with a given movement and others were not. The other question tackled indicated the relationship of the movements to other areas of knowledge, and this may be seen as a first attempt to indicate a theory of meaning[1] associated with each movement. Thus one could interpret Frege's logicism as indicating that appreciation of the meaning of mathematical propositions, includes the understanding of logical propositions, but it does not assure one, that he will understand physics. Furthermore, the entities of Frege's logicism have been identified as 'real', in the sense of being independent of the mind, but not real in the physical sense. One's picture of each movement is being clarified, and it is now possible to move each a bit more into focus.

1) <u>Logicism</u>

Both Frege and Russell considered arithmetic as a system of logical truths.

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1. Much has already been made of the realist/anti-realist dichotomy and it may be sufficient here to indicate two approaches to a theory of mathematical meaning roughly corresponding to these two sides. Thus, I know the meaning of p if 1) I can recognise a proof of p when given, and meaning must be linked to 'doing'; or 2) I know what has to be the case, for it to be true, but I do not actually have a proof here and now. Thus meaning is tied to truth conditions. A general point: meaning in mathematics does not generate the same set of problems as ordinary language, for it only has assertions.

a) Definition[1] of Mathematical Truth. With Platonic realism underpinning his view, Frege sees 'truth' and 'falsity' rather as abstract objects to which true and false propositions refer.[2] Thus, there is the reference of all mathematical propositions to truth or falsity, and the correspondence of arithmetic propositions to abstract reality. From Russell's standpoint there is an ultimate correspondence between mathematical propositions as abstractions and empirical reality. Given the attachment of both men to 'realism' (in the sense identified in Appendix, p. 278) one can recognise mathematical truth as defined in terms of something external to what any person may think or write.

Clearly, as the work of Goedel and Tarski has been produced after that of Frege and Russell, they might now have tried to employ a Semantic Theory of Truth with modifications to allow for the difficulties of consistency within their formalised system.

b) Criterion of Mathematical Truth. In order to see how Frege revised Leibniz's 'truths of reasoning' while retaining great sympathy for his overall programme, it is necessary to give some account of their differing views on 'truths of reasoning'.

For Leibniz, that a proposition was a 'truth of reasoning' rather than one of fact meant that the proposition could be rewritten through logical steps until it was glaringly[3] obvious that the two expressions 'bachelor' and 'unmarried man' either are or are not identical, or that the one expression is totally contained in the other. Thus in the case of 'a bachelor is an unmarried man' this becomes 'an unmarried man is an

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1. Where possible the discussion of mathematical truth is separated under the features noted from Russell, p. 74 above; those of a) 'definition', and b) 'criterion'.

2. For Frege there are sentences which are neither true nor false and their reference is what they state.

3. The reduction is to 'identical propositions whose opposite contains an express contradiction' (Monadology, para. 35). One might suggest that by 'express' is meant 'glaringly obvious'.

unmarried man'. In Leibniz's theory the following would be also truths of reasoning: 'Sulphuric acid contains Hydrogen' because in the language of chemistry it becomes by logical steps, 'H_2SO_4 contains H'. Similarly in the language of arithmetic '2 + 2 = 1 + 3' becomes '4 = 4'. Where there is not this possibility of rewriting the proposition by logical steps, the proposition would be treated as a 'truth of fact' and one would look to the world to show that it is true. Thus in the case of 'Essex beat Surrey last Saturday' one would require empirical evidence, for it is not as a result of the language of cricket that one could derive the identity, 'Essex is the side that beat Surrey last Saturday' and so reach, 'The side that beat Surrey last Saturday beat Surrey last Saturday'. However such rules do exist for God, according to Leibniz, for God's definition of Essex includes everything that they will ever do, including beating Surrey last Saturday. A truth of reason is thus one that cannot ever be shown to have had an error, while a truth of fact may be shown to possess such an error.

Frege reiterates Leibniz's position on the analytic a priori status of arithmetic, but provides a further modification to the meaning of 'analytic'. Frege asserts that an 'analytic proposition' is either deduced from definitions according to principles of logic or is such a principle. Thus, 'a bachelor is an unmarried man' becomes '$p \rightarrow p$ and 'sulphuric acid contains hydrogen' becomes '$p \wedge q \wedge r \rightarrow p$' and '2 + 2 = 1 + 3' becomes

$$2 = 1 + 1 \quad \text{and} \quad 3 = (1 + 1) + 1 \qquad \text{(Peano definition No. 10[1])}$$
$$(1 + 1) + (1 + 1) = (1 + 1) + 1 + 1$$
$$1 + 1 + 1 + 1 = 1 + 1 + 1 + 1$$
$$4 = 4$$

This is a logically tight view of what can be 'analytic', and it fits

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1. Peano's system is employed here because it is more straightforward than Frege's original form. To see both in action, From Frege to Goedel, pp. 30-97.

well with Frege's rejection of the idea that geometry could be seen as analytic. He treats geometric propositions as 'synthetic a priori', but hardly in Kant's sense, for while the foundation of their deductive systems is not logic, it is not the preconditions of human experience either. Frege is working after it had been shown that Euclidean geometry was not the only possible geometry, and so knew that not all the axioms could be derived from logical truths alone. He therefore holds the position that once the axioms of a geometric system are given, all else follows logically and in that sense is then independent of reality, a priori.

Thus arithmetic propositions are taken as 'analytic a priori' and so the conception of their truth is closely linked to this status. That is to say that a criterion of truth, 'being deduced from logical principles and definitions according to rules of logical inference is applicable to these propositions.

2) Formalism

a) Definition of Mathematical Truth. It is not easy to find Formalist definitions of truth, for the focus of this movement is to be found in the on-going work of pure mathematicians, contented with the success of their formal systems, rather than with philosophers to whom a definition might seem essential. However, Curry does clearly give the Correspondence Theory as the only option open to Formalists. As Curry says, mathematics is 'a body of propositions dealing with a certain subject matter; and these propositions are true insofar as they correspond with the facts.' ('Remarks on the Definition and Nature of Mathematics' in Philosophy of Mathematics, p. 153).

b) Criterion of Mathematical Truth. Fundamentally, Formalists wish to have a guarantee that their manipulations within a formal system are free from error. Thus Hilbert saw the deriving of a new proposition in a finite number of steps, from the basic axioms of the formal system, just in this light. Prior to Goedel, the Formalists believed that this formal derivability was the criterion of mathematical truth, and in no other way would mathematical truths occur. Once Goedel had shown that it was not an adequate criterion of mathematical truth, in the sense that some inconsistency may arise in any non-finite system, then one must accept the kind of modification that one finds in Putnam.[1] Putnam argues that one can still keep close to Hilbert's original programme, except that one must take the point of view that one treats 'mathematical axiom sets as "presumably consistent"' ('Mathematics without Foundations' in Mathematics, Matter and Method, p. 42). Thus Putnam links the criterion of derivability to the attitude that 'in general, and in the long run, true ideas are the ones that succeed'. ('The "Corroboration" of Theories', op. cit., p. 269). Thus 'derivability within a formal system' remains the criterion of mathematical truth for the Formalists, in practice.

3) Intuitionism

The Intuitionist view of mathematical truth is heavily grounded in Kant's introduction of the notion of synthetic *a priori* propositions. Some explanation of how such a category arose will have to be given before one can clarify the more contemporary position of Intuitionism.

Kant largely accepts Leibniz's characterisation of distinguishing truths of reason from truths of fact, but it will be seen that Kant does modify the original position. He divides all propositions into three groups and not two as Leibniz had done. In addition to 'analytic *a priori*' propositions like 'a bachelor is an unmarried man', and 'synthetic

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1. There is an alternative direction possible as a response to Goedel, and this path has been chosen by Carnap who has become more and more openly committed to Platonic realism. In this way one can argue that the only mathematical truths are those that correspond to this reality, even if the reality is no more than a further language form at a higher level of universality. (Compare Carnap's article, 'The Logicist Foundations of Mathematics', given in 1931, reprinted in Philosophy of Mathematics, pp. 31-41, with his article, 'Empiricism, Semantics, and Ontology' in Meaning and Necessity, 1955 (2nd Edition)).

a posteriori' propositions like 'Essex beat Surrey last Saturday' there are 'synthetic a priori' propositions like '2 + 2 = 1 + 3' which Leibniz had included in 'truths of reasoning'.

Kant argues that there are two types of 'synthetic a priori' propositions.

a) Those related to the spatio-temporal framework that makes any experience of the phenomenal world possible, e.g. 'What is here cannot also be there.'

b) Those related to the categorial principles, not just of mathematics, but also of the natural sciences and metaphysics, which make knowledge of the phenomenal world possible, e.g. 'Every event has a cause.'

Kant argues that Leibniz has not recognised that there are these propositions that do not merely include linguistic constructs that could be turned out by a '*characteristica universalis*',[1] but if they are mathematical for example, they are the constructs of space and time. One must go beyond linguistic constructs to intuitions corresponding to the individual elements in space and time. Yet if they are so closely tied to the real world, then one may ask why knowledge of them does not come *a posteriori*. Kant's argument is that such knowledge relates to changes in the material objects of the world, and what he is identifying is precisely that knowledge about the features of the phenomenal world, which are imposed by any knowing subject upon his experiences.

The conclusion seems to be that Kant has not just introduced the notion of 'synthetic', but modified those of '*a priori*' and '*a posteriori*' also. Thus *a priori* knowledge makes no requirement on one to make

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1. Leibniz's argument is that mathematics is indistinguishable from logic, for both can be reduced to this mechanical system (see p. 24 above). While Leibniz is committed to mathematics being true in all possible worlds, Kant is not forced by the logic of his position to propose that mathematics as existent was *the* mathematics, but only that it is *the* mathematics for *this* world.

empirical investigations to understand it, not because it is necessarily disconnected from the world, but only because it is necessarily disconnected from changes in the world. Similarly, *a posteriori* knowledge makes requirements about changes in the world for it to be understood, or at least what is logically possible to be changing. A 'synthetic *a priori* proposition' is thereby one that identifies constructs constitutive of invariant features of the real world and whose truth depends ultimately upon 'constructibility' rather than 'definitions'. This is the clear starting point of Intuitionism.

For the Intuitionists there is no external or eternal reality to which mathematical propositions can correspond and in fact mathematics is for ever under construction and reinterpretation. This activity occurs in people's minds and as such is difficult to identify with any of the criteria of mathematical truth already discussed.[1]

a) Definition of Mathematical Truth for Intuitionistic Logic can be shown compatible with an appropriately modified Semantic Theory of Truth. Both Dummett ('The Philosophical Basis of Intuitionistic Logic' in Dummett) and Beth (*The Foundations of Mathematics*, pp. 442-62) have shown that provided the notion of 'proof' is construed intuitionistically, then 'It is true that 'p' iff p' can be employed as the definition of truth. Thus for mathematical truth 'p' represents an intuitionist computation. In other words, there is the room for a compromise position in the philosophy of mathematics across the realist/anti-realist gulf. Signs of such a bridge are to be found in Wittgenstein,[2] for he did question the absoluteness of bivalence, and does look for a third position

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1. Intuitionism is not readily linked in this respect with any of the theories of truth outlined in the Appendix, either.

2. It was argued in Chapter 3 above that Wittgenstein fits most readily as a bridge between the Intuitionist and Hypothetical movements, but with a leaning towards the former. Particularly, Wittgenstein emphasises 'constructibility' within mathematics, as the Intuitionists do. Thus he sees mathematics as the construction of rules in a language where 'constructed sentences' and 'construction rules' are one.

distinct from platonic reality and mental constructs for the conception of mathematical truth. Wittgenstein supported the public form of constructibility as one might interpret Intuitionistic logic on one side, but also the playing of language-games on the other. Although the Wittgenstein of the Remarks renounces the Correspondence Theory of Truth for a Redundancy Theory, the gap between this latter theory and Tarski's, for formal languages, is very narrow. Ramsey's notion of 'It is true that 'p' iff p' being reduced to 'p' iff p is very close to the foundation stone of Tarski's theory. The closeness can be encapsulated in a simple example. If one asks, 'is 3 + 4 = 7?' then what one needs to know is whether or not the rule '3 + 4 = 7' holds for mathematics, i.e. 'p' iff p. Thus Wittgenstein came to argue that mathematical truth can only be presented according to a correspondence theory, in a peculiar sense of that theory, for '3 + 4 = 7' is both the true proposition and the rule to which it is thought to correspond. There is neither a separated reality nor the need for a hierarchical structure in mathematics, for empirical reality enters for the use of proven mathematics and not before.

Similarly, one is left with the realisation that at the very centre of Intuitionism is actual construction, and any definition of mathematical truth must involve this. This is not identifiable in definitions that rely upon existent, abstract or empirical objects, as any theory, employing correspondence for its definition of truth, does.

b) <u>Criterion of Mathematical Truth</u>. Some clue to such a criterion may be given by some initial notions found in Kant's <u>Introduction to Logic</u> (pp. 13-14). Having identified knowledge from principles as <u>a priori</u>, Kant goes on,

> Mathematics...is <u>Rational knowledge from the construction of concepts</u>. We <u>construct</u> concepts when we present them in intuition <u>a priori</u>... In mathematics we use Reason <u>in concrete</u>; the intuition, however is not empirical, but we make something <u>a priori</u> the object of the intuition. [Underlinings occur as italics in the translation]

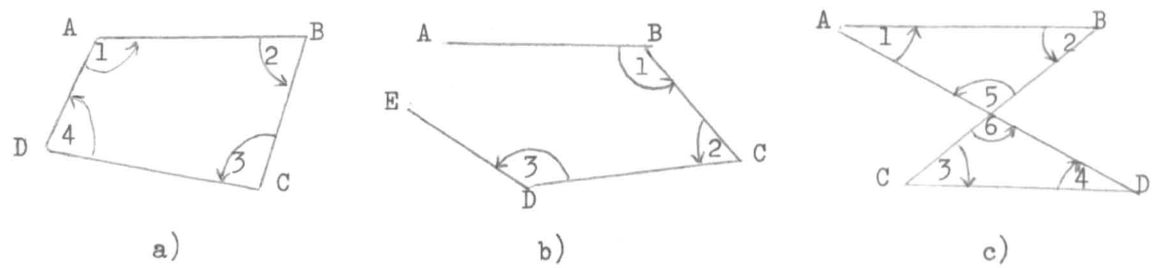
'Construction' is thereby the means for demonstrating both what is and what is not and simultaneously what is true and what is false. However, there is the linguistic representation of mathematics to be found through Intuitionistic logic, and any proposition of this logic only comes into existence because there is a corresponding mathematical construction. The mathematics justifies the logic rather than the logic justifying the mathematics as in Logicism. In both the thinking and its linguistic representation, the truth of propositions is related to successful production. In the former, truth is a construction completed, and in the latter, a newly derived formula. The important metaphysical point is the argument that something is true or false only if it is shown to be so. This separates Intuitionism quite strictly from Formalism and Logicism. Dummett has argued that there is no logical reason why one should not take this principle into languages, where it had not been previously accepted, unless the language includes a metaphysical commitment to all propositions being either true or false. That is, to hold some form of 'realism'. One would be holding that in some sense all true propositions are already identified. (See Dummett, <u>op. cit</u>., pp. xxxix, xl, xli and the article 'The Philosophical Basis of Intuitionistic Logic').

4) <u>The Hypothetical</u>

In the area of mathematical truth, there would seem to be a short step from the work of Curry and of Wittgenstein to that of Lakatos. All three see 'proof' as a more central term than 'truth' in mathematics. However Curry has reached there less willingly than Wittgenstein and certainly Lakatos. To Lakatos mathematical proofs are open to refutation just as proofs in any other science.

<u>Lakatos</u> argues that one never defines a concept as tightly as one hopes. For example, 'All quadrilaterals have four angles' seems self-evidently analytic but Lakatos would question whether the meaning of

quadrilateral necessarily rules out figures with three angles or six angles, as in b) and c) below.



The question that arises for Lakatos is when does this 'monster-barring' or 'concept-stretching' occur, and when is it valid. He argues that acceptance or rejection of such modifications is dependent upon agreement within the community using this language, for example mathematicians. Given Lakatos' position, 'a bachelor is an unmarried man' is true only if agreement has been reached about monsters. Quine identifies this as 'socially stimulus analytic'. This becomes logically no different from certain rules being agreed, before one can say that 'Essex beat Surrey last Saturday', 'socially stimulus synthetic'.

<u>Quine</u> believes that even if an adequate distinction is provided, one will find that the two classes identified are in fact empty. This is the position that he first proposed in 'Two Dogmas of Empiricism'. Quine argues that 'analytic' is defined in terms of 'synonymy', and even if one were not worried about 'the notion of "synonymy" (which) is no less in need of clarification than analyticity itself' ('Two Dogmas of Empiricism' in <u>From a Logical Point of View</u>, p. 23), one could still never feel secure in a logical sense that two concepts were synonymous. In other words, just how certain one is that 'bachelor' is indistinguishable from 'unmarried man' is a psychological rather than a logical state. Even if one accepts uncertainty here, Quine argues that one could not know how many exceptions one would need to identify of the kind,

'Bachelor has less than ten letters', for surely if they were interchangeable[1] in Leibniz's sense, then 'unmarried man' ought to fit here too. There have been many defences of Quine's attacks, and typical of these is Cooper's in Philosophy and the Nature of Language. He argues that this particular attack of Quine's is invalid because the expressions are not being used semantically in this example. In other words, interchangeability refers only to situations where meanings are involved.

However Quine still has the argument that analytic propositions only carry weight, if one can identify that they are purely dependent upon linguistic as against factual considerations. Quine argues that this absence of factual considerations cannot be shown logically to be necessarily the case. Since this article[2] Quine seems to have accepted a slightly weaker position, for in Word and Object for example, logical truths are for all intent and purposes identified as analytic (See Word and Object, pp. 54-67).

While, like Quine, Lakatos rejects the validity of the analytic/synthetic distinction, there is no certainty that he[3] accepts, like Quine, that 'deductive, inferential intuition is infallible' (Lakatos, Proofs and Refutations, p. 138). Even if Lakatos accepted a special status for logical truths, he would want it recognised that these are a small minority of the principles that make up all reasoning, and are swamped by all the mathematical and heuristic procedures.

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1. Quine sees 'interchangeability' as having the same kinds of problems as synonymy. Leibniz appends the phrase '*salva veritate*' to the notion of substitution, but if one considers this for the example given, it fails. Interchangeability does not occur without a change in the truth value of the given proposition. ('Two Dogmas of Empiricism' in From a Logical Point of View, pp. 27-32).

2. 'Two Dogmas of Empiricism'.

3. The question of how much of logical form Lakatos does wish to retain was discussed on p. 68 above. This unresolvable debate has no implications for Lakatos' clear acceptance that 'formal propositions', which have no meaning, are '*a priori*' truths; that is, logical truths. These are not refutable. Quine's view of logical truths was discussed briefly on p. 37 above.

Lakatos holds an evolutionary view of all knowledge, including mathematical. A theorem taken as true/proven at any one time is so, relative to a particular system and any attack on the theorem may be of two kinds. Either one tries to refute the conjecture within the system, or one demonstrates the total inadequacy of the system and then provides the same or modified conjecture translated into a new system. To say that a proposition is true, is not to say that it has withstood 'the method of science' as Peirce said, but 'the method of proofs and refutations'. Lakatos' position has a strong tinge of pragmatism in both this respect and in the evolutionary concern (see Appendix). Thus the links between the hypothetical movement and the pragmatists suggested in Chapter 4 are reinforced here.

Like Lakatos, Ormell accepts that in some circumstances mathematics may be treated as Curry views it, an 'uninterpreted language/symbol game' ('Mathematics, Science of Possibility', in _Int. J. Math. Educ. Sci. Technol_. 1972), but this should be an ever-decreasing part of the picture.[1] That is to say, the main view should be of interpreted mathematics, the 'potential model expressions'. This is to put the emphasis of mathematics upon problem-solving as Lakatos and Dewey do. Ormell shows reservations reminiscent of Wittgenstein, and stresses the 'symbol game' side of mathematics as well as the hypothetical, but always linked to use. This seems to be indicative of a position combining Formalism and Pragmatism, rather as Peirce linked Kantian presuppositions to Pragmatic maxims. The result for Ormell is a view of mathematical propositions as synthetic _a posteriori_, where the verification occurs through all relevant

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1. There are some obvious similarities between Putnam's view of mathematics and Ormell's. Putnam discusses 'the use of quasi-empirical methods in mathematics' and Ormell of 'quasi-apparatus'. Similarly on the 'final collapse of the _a priori_' Putnam believes that 'much of mathematics too is "empirical"'. However Putnam remains a hard realist who would not accept Wittgenstein's waverings. See _Mathematics, Matter and Method_, 'What is Mathematical Truth'.

participators being trained to carry out the manipulations in the same way. As Ormell says, 'In the end we know elementary arithmetic only because we, our contemporaries and ancestors have found it so.' As with Wittgenstein this leaves truth somewhat redundant, for words like 'warranted', 'correct procedure', 'proved' seem to cover the same ground, though not necessarily by themselves.

CONCLUSION

a) Definition of Mathematical Truth. Given the concern of the earlier chapters with the ontological positions of the movements introduced there, one is left wondering whether the ontological commitment dictates or simply identifies what form the definition of truth will have. Thus, if a movement asserts the external existence of mathematical objects, then one may immediately identify it as holding/going to hold some form of Correspondence Theory. Similarly the anti-realists may be thought to necessarily link their undetermined ontology with fully determined proofs, for that in a sense, is how their mathematics is born. For this reason and because of the rejection of bivalence in Intuitionism, one must be cautious in suggesting that all four movements have a sufficiently formalised part or total structure, that is able to meet Tarski's requirements for a definition of truth for their 'language'.

b) Criteria of Mathematical Truth. In this chapter a common feature of three movements has been the derivability of true propositions from some set of axioms. The exception is Intuitionism in which constructibility replaces derivability, and whatever set of axioms is identified as basic within some Intuitionistic logic, it represents a possible interpretation of mental activity and is not itself the foundation of mathematics. While all other movements presume a logic exists before mathematics develops, the Intuitionist argues that the logic comes into being only as a result of the development of mathematics.

More disparity occurs when one looks for other criteria of truth, rather than a definition. Then one sees that derivability is insufficient for Putnam, Lakatos and Ormell. Their unhappiness is combined with a more 'empirical' view of mathematics. That is to identify one split between *a priori* and non-*a priori* movements - Logicists and Intuitionists, opposed to Formalists and Hypothesisors. In fact Lakatos and Ormell give every indication that they agree with Quine on the essential redundancy of the analytic/synthetic distinction.

This conception may be put in slightly different terms, if one asks what account each movement would give of the objectivity of mathematics. Under what circumstances would each movement identify a proposition S as true? The Logicist would assert that the deductive proof of S from logical truths and by logical principles would show that it was true. The Formalist would look to a proof too but this would be based on axioms intuited from the world and not from logic, but obeying logical principles in the manipulation. The Hypothesisor rejects the strict Formalist's proof as only half a proof, for neither has it stood the public scrutiny of proof-analysis, nor has it been shown to be of applicative value either within or without mathematics, at least potentially. Finally, the Intuitionist may agree that this linguistic interpretation of some mental construction identified as S, may in this linguistic form be checked by a proof comparable to the Formalist one, but with respect to the principles of Intuitionistic logic, not Classical logic. The enlightened Intuitionist may go further and say that clearly by 'objectivity' one means the proof of S, originally mentally constructed, being checked in Intuitionistic logic.

CONCLUSION TO PART 1

INTRODUCTION

In the first part of this thesis four movements in the philosophy of mathematics have been outlined. Three of these movements have a tradition of being demarcated as such. These are Logicism, Formalism and Intuitionism. A fourth movement has been constructed by the author, bringing together nineteenth century Pragmatism, twentieth century Falsificationism and the contemporary views of a director of an existent Schools Council Mathematics Project. This movement is called 'the Hypothetical' and is unified particularly by the general agreement that mathematics is used in problem-solving through 'thought-experimentation', rather than through 'laboratory-experimentation'. It is in this sense, 'hypothetical'.

These movements have been selected because they have had direct or indirect impact on the ways mathematics has been and is being taught in secondary schools. In order to give a clear account of each of these movements, the thesis begins with a brief resume of the views on mathematics of Plato, Aristotle, Leibniz and Kant, who are identified as forefathers of the movements central to this thesis. To ensure clarity of exposition and unity in this first part of the thesis, three questions have been identified and the body of the first part of the thesis consists in answering these questions:

1) Are there eternal objects of mathematics?
2) Is 'mathematics' logically distinct from the 'empirical sciences'?
3) What is identified by the phrase 'mathematical truth' and how is 'mathematical truth' demonstrated?

QUESTION ONE is identifiably concerned with the ontology of mathematics. The first two movements discussed were Logicism and Formalism,[1] and are both committed to some form of ontological realism. That is, mathematics does not consist of mental entities, depending exclusively on man for their existence. However, it is only the Platonic realism of Plato himself, and of Frege, that assuredly can be said to present the objects of mathematics as eternal. The other Logicists and Formalists link mathematical objects, not to man alone, but to man and the existence of some physical reality, a real world. In other words, only realistic logicism gives an unrestrained positive response to this first question. The term 'eternal' bars all other philosophers.

The Intuitionists take the rejection further. They see mathematics as a solely mental activity whose public form, as represented by intuitionistic logic, can only have a contingent relationship with mathematics as it is in itself. That is to say, the written form of mathematics presents a description of an ideal system. Consequently, there may be several attempts of differing kinds to represent mathematics publicly.[2] Finally, the Hypothesisors generally hold a view similar to that held by the Formalists that mathematical existence is a contingent state, dependent upon the public performances and communications of men.

Clearly, the differing ontological standpoints can direct one's teaching in different directions, and this point is developed in part 2 of the thesis. A teacher influenced by Platonism can argue that it makes sense to talk of 'uncovering' the eternal beauty of mathematics, but this can be no more than a stretched metaphor for a teacher influenced by Intuitionism. He is much more likely to stress the value of studying mathematics for developing the mind, for mathematics to him, as 'mental constructibility', is a way of reasoning.

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1. Chapter 2, pp. 27 to 36.
2. Page 46 above.

QUESTION TWO requires a clarification of the logical status of mathematics, in relation to other areas considered to be key features of education. This question is particularly appropriate at a time when there is so much public concern with the 'needs of society' and the production of scientifically aware youngsters.[1]

This century began with Logicists claiming that mathematics was not just logically different from the empirical sciences, but reducible to logic. This position is contrasted in this first part of the thesis with the Intuitionist view, that mathematics is autonomous, and the view of Formalists and Hypothesisors that it is a science, but a special science. Formalists see mathematics as 'the science of formal systems' and the Hypothetical movement see it as 'the science of possibility'. In both these cases, mathematics stands logically apart from other sciences, because for Formalists it does not refer to any natural particulars in the real world, and for Hypothesisors it is seen as describing 'what can be', and not 'what is'.

Thus every movement identifies some quality of mathematics, by which one could come to argue that its study gives a dimension to one's education that is not obtainable through the empirical sciences alone. The extent to which this can be combined with other factors to form a coherent policy for teaching mathematics will be considered in the following parts of the thesis.

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1. Aspects of Secondary Education in England. 'Problems demanding the everyday use of language were receiving insufficient attention; and computation was practised..., unrelated to practical questions in which it is necessary to subject the calculations to commonsense checks' (p. 141, ch. 7, para. 6.18). 'The ideas of the courses need to be shown in a broader setting,...Mathematics was too little related to the world outside and to other subjects in the school,...' (p. 159, ch. 7, para. 10.14). Mathematics is being asked to go, where H.M.Inspectors believe language teaching now is.

QUESTION THREE. As the majority of movements follow the path of ontological realism, so they also follow the path of epistemological realism. Only in the work of Lakatos is there any explicit questioning of whether or not the truths of mathematics are held relative to social contexts.[1] The other Hypothesisors identified, Peirce and Ormell, seem to hold the view that the solutions to problems that are achieved by man, may be fallible, yet man can never know when he has achieved the right solution, although such solutions may be obtained. Thus Ormell in particular comes close to accepting the redundancy of 'truth', in favour of terms like 'warranted assertion'. There are signs that Lakatos takes the next step and sees 'mathematical truth' as a relative concept. Such relativism has explicit proponents who are not discussed in this thesis. Their[2] omission is justified because they have as yet to make a significant mark on the mathematics education of England and Wales. Furthermore, the central argument of this thesis will be seen to be unaffected by the omission. It is sufficient for this argument that question three is shown to have a variety of answers, even if some more extreme responses are omitted.

In part 2 of the thesis the variety of responses is seen reflected in the differing approaches presented for teaching mathematics. Thus, a teacher influenced by strict Formalism may consider a formally derived proof of $(a - b)(a + b) = (a^2 - b^2)$, as adequate evidence that a student has shown that a given mathematical proposition holds, while a teacher influenced by the Hypothetical movement may also require the student to

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1. This is found in the dispute identified on p. 87 above.
2. In philosophy of science this standpoint is well developed in P. K. Feyerabend, Against Method. In mathematics education the main force in this direction is W. M. Brookes but he rarely publishes his views although the influence is there in statements like the following, 'The nature and content of mathematics is an element of the culture of a people and liable to change with time. Any definition of mathematics is a personal statement fixed in time.' (Notes on mathematics for children, A.T.M., p. 143).

show some internal or external application for the expression, as occurs when asked to calculate $75^2 - 25^2$.

CONCLUSION

At this point four movements have been identified through their distinct views of the nature of mathematics. In the second part of the thesis these distinctions will be drawn upon to discuss the differing approaches to mathematics teaching to be found in secondary schools, from the overriding concern for 'mathematical beauty and form' to the constant regard for 'practical pay-off'.

PART 2: MATHEMATICS TEACHING

C H A P T E R 6

SEARCH FOR A COMMON FRAMEWORK

INTRODUCTION

In the first part of the thesis four perspectives on mathematics are outlined but no attempt is made to show that one movement provides a more accurate description than another. Given that the central focus of this thesis is mathematics education it would be short-sighted to exclude one or more movements if they have an on-going influence in mathematics education. The main aim of this part of the thesis is to highlight the influences present in mathematics teaching, which are due to philosophy of mathematics, rather than to assert what ought to be present in mathematics teaching.[1]

There is a great temptation to attach to each philosophical movement outlined in part 1, a distinct perspective upon mathematics teaching. Thus for example, one would expect a teacher who admits to a Formalist philosophy, believing like Curry that mathematics consists solely of the manipulation of symbols, to teach mathematics as a 'Game'.[2] It is clear that such tight logical connections cannot be drawn when one discusses the complexities of successful teaching. The teacher may choose to do what will motivate his class rather than what will connect immediately to his long-term objective, of showing that mathematics rests on Formalist foundations.

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1. In part 3 of the thesis some prescriptions are made.
2. The linking of 'mathematics' with 'games' is first discussed on pp.

Nevertheless it would seem worth identifying movements in Mathematics Teaching in order to provide a clear structure to this part of the thesis, as there was in the first part. In the final pages of part 1 it is made clear that for each philosophy of mathematics there is a distinctive view of how mathematics is enlarged as a publicly accepted body of knowledge. These views are:

1) The Logicist view considers the enlarging of mathematics as an exercise in uncovering what one knows to be pre-existent.
2) The Formalist view considers this as an exercise in discovering new ways of combining what is independently given.
3) The Intuitionist view considers this as an exercise in creating totally new things.
4) The Hypothesisor view considers this as an exercise in inventing structures that have a potential usefulness.

In this part of the thesis four teaching perspectives are identified and it will be argued that there are existent movements in mathematics education, corresponding to each of them. For example, Nuffield mathematics has a commitment to 'discovery methods' and sets the child out to uncover wonderful new ways to look at the world.[1] The four teaching perspectives are,

1) The Teaching of Mathematics as Aesthetically-orientated.
2) The Teaching of Mathematics as a Game (or Family of Games).
3) The Teaching of Mathematics as a member of the Natural Sciences.
4) The Teaching of Mathematics as Technologically-orientated.

The order is seen predominantly as a 'Culture-continuum' rather than corresponding to the order of philosophical movements discussed in part 1. It

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1. As the originator G. Matthews says of Nuffield Mathematics, its objective is 'to help the children develop gradually...from discovery with things to eventual abstraction with pencil and paper' ('Mathematics in the Middle Years', S.C.W.P. No. 22, p. 64).

reflects a personal view of how many people see the curriculum, ranging from the least to the most 'useful'. However a fifth 'perspective' is presented as somewhat of a radical alternative. This is the view that as far as possible all teaching is best done in an interdisciplinary context. This part of the thesis ends with a discussion of the implications of such a viewpoint. It is important to remember that the focus has moved from philosophical views of mathematics to that of mathematics educators generally aware of philosophy. The two foci are inter-related, because one's beliefs about the nature of mathematics will put constraints on the way one teaches.

Over the last twenty years there have definitely been radical changes in all education and when one mentions a particular teaching approach it may well depend on one's age whether one imagines it going on in the Infant School or in a class for M.Sc. students. Anyone looking in S.M.P. Book 1[1] and then in the Schools Council Curriculum Bulletin No. 1 will notice that a topic, say tessellations, is linked much more to the language of experimentation in the latter, more recently revised book. It is doubtful if such clear divides would have happened when they did, if there had not been in the previous fifty years major divergences of opinion in mathematics, the natural and social sciences and in philosophical reflections upon them. The result is that in this thesis one has been able to identify four philosophies of mathematics all founded within the last hundred years and can now go on to consider a range of teaching perspectives, using examples all being employed in schools today. While a particular philosophy does not commit one irreconcilably to a particular perspective, some links seem more natural than others.

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1. S.M.P. is the School Mathematics Project. With the Midlands Mathematics Experiment it was the first of what have become known as 'Modern Mathematics' texts but even in this area there has continued to be changes from 1962 when S.M.P. Book 1 was published to 1972 when this Bulletin was revised.

THE PERSPECTIVES

The differing perspectives will be described as employed by a fictitious 'young teacher'. He is a mathematics graduate who was told at Primary School what fun it is to do 'sums', treating mathematics very much as a 'game'. This approach contrasted sharply with the more utilitarian perspective suggested as essential for someone hoping to enter a Grammar School. Once in the Grammar School the utilitarian viewpoint was reinforced by talk of Pure Mathematics being one 'O' level that one had to get to take up a professional career, particularly in Accountancy, Economics and Law as well as being compulsory for studying science at a 'good university'. This 'young teacher' came to find intrinsic pleasure in learning mathematics, happy to study it for its own sake. Fortunately this was reinforced by his 'A' level Pure Mathematics teacher who saw 'mathematics as an art', an area of 'creative thinking'.[1] At the same time his physics teacher made it equally clear that mathematics was really no different from physics and the other sciences, except that it was far less exciting and considerably less useful. However the links with 'Art' and 'Games' were reinforced at university by Pure Mathematicians, but then connections to be drawn with the scientific method were refreshed by tutors on his Graduate Certificate Course. There the focus was on 'Discovery Teaching Methods'. These practical influences of teacher education came just as the young teacher had begun to think that he had acquired 'mathematical taste' as Zeeman identifies it. He seemed to recognise the subtle qualities of 'elegance, intrinsic beauty, profundity, generality, simplicity, depth, subtlety and economy' in many parts of Mathematics. Thus in his attempt to satisfy this range of acceptable influences this 'young teacher'

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1. See support for this approach in Zeeman E. C., 'Mathematics and Creative Thinking' in Mathematics in School, 2, 305 (1972). He defines 'elegance, intrinsic beauty, profundity...' as found in all aesthetic appreciation.

comes to employ differing perspectives with different classes in his first teaching post, at an 11-18 Comprehensive.

1) The Teaching of Mathematics as Aesthetically-orientated

The young teacher has been given syllabuses and text books to guide him and he begins with preparation of his 'A' level teaching. He hopes that his students will come to value mathematics 'in all its beauty' as he has done. He is to introduce them to co-ordinate geometry, say. His first objective is that they should appreciate the elegance and rigour of the proofs in this area of mathematics. He may have lots of objectives so let one more be noted in order to make clearer the characteristics of this approach. The students will have the opportunity to 'create'[1] proofs of their own and to compare them with other proofs, and to see which are 'better' than which with respect to parsimony, elegance, et al. In this way the students will recognise that aesthetic terminology can be meaningfully linked to works of mathematics, just as one would use them in critically appraising a painting, say. Thus the approach shows itself in two ways. Firstly, the students are encouraged to develop 'mathematical taste' and demonstrate their understanding in using critical language and even showing preferences among different proofs.[2] Secondly, the students are led through proofs and then left to their own devices to 'create' inverses and corollaries.[3]

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1. See the discussion of 'creativity' within educational objectives in Elliott, R. K. 'The Concept of Creativity', Proc. Phil. Educ. Soc. of Gt. Brit. V, 1, 97-104 (1971) and 'Versions of Creativity', ibid., V, 2, 139-52 (1971), Ormell, C. P. 'Bloom's taxonomy and the objectives of education', Educ. Res. 17, 3-18, and in the next chapter.

2. One student prefers to use trigonometric proofs rather than Euclidean ones wherever possible because of their brevity, but another student likes the subtlety of constructions in Euclidean proofs even if they are many times as long.

3. Given Pythagoras theorem that 'In a right angled triangle the area of the square on the hypotenuse is equal to the areas on the other two sides' then the inverse is 'If the area of the hypotenuse is equal to...then the triangle is right-angled'. A corollary would be any proof following directly from the proof of Pythagoras, e.g. 'The centre of a circle lies on the perpendicular bisector of any chord'.

2) The Teaching of Mathematics as a Game

The young teacher considers next his youngest pupils, eleven year-olds. He decides to start with 'co-ordinates' and to introduce this through 'Battleships'[1] which many of the class may know how to play. Battleships is a game, with rules covering every allowable activity within the game and also procedures for deciding when it has ended and who has won. The teacher is encouraging the pupils to view mathematics as a family of games. Each part of mathematics is to be seen as possessing definite rules for everything one is allowed to do and procedures by which one can recognise that one has completed some 'sub-game', e.g. proved a theorem. So for these pupils the concept of 'games-teaching' may be said to be gaining a new dimension. However these games are not intended to be competitive but are comparable to patience at cards or playing against the course, at golf. 'Winning' is the successful completion of a 'sub-game', like proving a theorem or doing ten sums, checking them and finding that they are correct.

Even though the teacher chose this approach for this class, he may at the same time introduce the language of 'aesthetic appraisal' (not everything that is 'beautiful' is a work of art - a beautiful cover drive). He may want also to indicate that battleship lay-outs are most inventive sometimes or that one pupil has 'created' a particularly complex solid or intricate symmetrical pattern. He may not feel that 'create' as used here is synonymous with its use referring to Blake as 'that creative genius', but it is still in his mind to begin to indicate the possible association of 'art' and 'mathematics'. The teacher is happy also for

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1. Battleships is a game for two where each person has a grid, numbered 1 to 8 horizontally and A to H vertically for example. On the grid he places ships, e.g. x submarines, y frigates and z destroyers where each type of ship covers a different number of squares. By calling the names of different squares to your opponent (C4, say and opponent says, 'Hit' or 'Miss') one eventually covers a second grid with the whereabouts of your opponent's ships.

a child to make a discovery. In class, the pupils having thought of a three digit number find someone who would like to try and work out what it is by guessing numbers in each of three places. There may be many other rules and most likely the work is in base 5 say, rather than base 10.[1] The point is that one or more children return the next day and say they have a game at home called 'Mastermind' and although it uses colours for numbers it really is the same game - that is a discovery. This perspective may be seen however as discouraging the pupils associating mathematics with real-life situations. The idea that mathematics is a game may facilitate transfer to the Darts board or Netball pitch rather than to the Science Laboratory or Home Economics base. There is certainly room for further discussion on this point in the coming chapters.

3) The Teaching of Mathematics as a member of the Natural Sciences

The young teacher's second years are an able group, an express band taking '0' level in four years. They are to start the year with a look at 'Euler's rule'[2] and the young teacher, remembering a desire for 'discovery methods', can see that it is very easy to present this as analogous to the discovery of a law in science. The students can look at pictures of polyhedra and make polyhedra and decide what constant features they have, like corners and so on. Then they try to make conjectures about these vertices (corners), faces and edges. If they are guided to produce a table of information then arriving at hypotheses may be facilitated. Hypotheses may be tested and modified until some form of Euler's rule appears and is generally accepted.

One may accept that the scientific method has been the basis of

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1. Ten is called the base of our number system. Place-value is determined by the base so we have 1's, 10's, 10 x 10's, ... If the base were five then there would be 1's, 5's, 5 x 5's, ... 34 base 10 is 3 x 10 + 4 x 1 but 34 base 5 is 3 x 5 + 4 x 1.

2. 'Euler's rule' shows a relationship for polyhedra (regular solids like cubes) such that F + V = E + 2. Faces - Vertices - Edges.

some mathematics learning and yet make it clear to the learner that mathematics is not generally thought to be inductivist. In Nuffield mathematics the child is encouraged to experiment and find the solution for concrete problems, but at the appropriate point of cognitive development he is encouraged to go on, to 'abstractions and generalisations'.[1] The child is not brought finally to believe that scientific induction is the method of mathematics but that it has a part to play in mathematical exploration. In this way the approach does not violate hard-held philosophical positions that a teacher may have. The experimental method can be accepted as excellent for motivation, provided the pupil is not given a view of mathematics that is so radically empiricist that he comes into conflict with nearly all teachers involved with mathematics, and is left with an overriding sceptism of mathematical results in later life. It may be valuable to show that mathematics must relate to science but one must be careful in explaining in what sense it is science.

4) <u>The Teaching of Mathematics as Technologically-orientated</u>

The young teacher found that the third year syllabus began with 'flow charts',[2] and he decided to begin with a few general knowledge-type problems. For example, the crossing of a river by soldiers who have a boat

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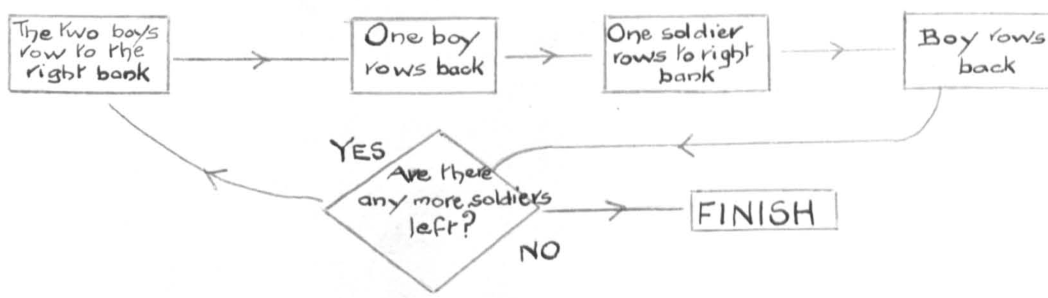
1. This division between mathematical exploration and mathematics not being identical to the other natural sciences in its method comes out clearly in the writings of those concerned with this approach. The book, <u>Mathematics Through School</u> contains the views of fifteen such mathematics educationists.

2. A flow chart indicates the possible sequence of procedures that could occur in a given situation, assuming that any question is answered 'yes' or 'no'. They facilitate the analysis of a problem. Thus a flow chart could help the solution of the above problem in the following way: (See <u>SMP Book F</u>, pp. 1-3).



owned by two boys, that takes one adult or two boys at a time. The teacher intends to show diagrammatic solutions of this and other such problems. He will use flow charts. The flow chart is a model of a solution for problems of a particular kind, and given information about a particular problem, aids its solution. Once the class become aware of the nature of flow charts, further flow charts will be employed in the solution of linear and quadratic equations, for example. The pupils learn how to construct for themselves kits of tools that are sometimes used to deal with problems purely within mathematics, like a linear equation, and at other times a kit may be used to solve a problem that the pupils would identify totally with real-life rather than connected to mathematics. In this comprehensive there is genuine concern about where is the best position for a new vending machine, to minimise congestion. Vending machines are heavy to move about but mathematical models[1] of the possibilities are not. Before the school makes a decision opportunities are given for debate. Differing models are investigated by the model-builders. Modifications may result but eventually an agreed best solution becomes established. These 'mental models' may be seen as comparable to the physical models that an engineer uses or the computer models of a country's economic growth that the Chancellor may use. Models are found throughout the applied, natural and social sciences. From this perspective, mathematics is to be seen as heavily concerned with applicability.

One expects a young teacher to want his pupils to learn as much about mathematics as their corresponding levels of conceptual development seem

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1. While the number bases defined in footnote 1, p. 102 may be seen as simply conventions to facilitate the procedures of mathematics, 'models' as identified here are manifestations of ideas at a level of abstraction further from the concrete than the original, e. g. Chess models the original Chinese game with real people. A linear graph may be presented as a model of a motor car's acceleration performance. In Teaching Mathematics Applicable - Introductory Guide (Heinemann, 1979) pp. 10-15 there is a brief but thoroughly exemplified account of 'modelling' and further elaboration occurs in Chapter 10 below.

to indicate is appropriate. This desire may be facilitated by the use of models. They can vary in levels of concreteness and complexity. Models are not able to illustrate all the features in the original and so the teacher may guide the pupil to a model at a level of interpretation at which he can work successfully. It is hoped also that the pupil will see in the model more than the means to solve one problem, namely the 'potential' to model other situations. Within the objectives of this technologically-orientated approach is the commitment to show as far as possible the relevance of knowledge acquired, not only to real-life problems but also to problems in other parts of mathematics itself, in the way that co-ordinate geometry[1] may be considered to model Euclidean. Inevitably the teacher sometimes stimulates the production of 'potential models' as a game, without identifying any particular applications at that time.

A further perspective may well have been expected. There has been much talk of 'mathematics as a language'[2] as a successful focus for mathematics teaching. Underlying this view is the analogy with the natural acquisition of a first language. It is assumed that a child could acquire 'naturally' a first 'numerical' language too. Teachers may want to encourage this view of mathematics as no more difficult to learn than one's mother tongue, but it does not seem to identify any one approach or view of what mathematics is. Mathematics can only be identified with a restricted use of 'language'. It is difficult to see how a set of symbols held under a tight syntactic structure and even capable of meeting the

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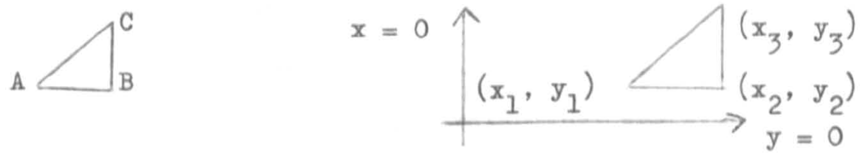
1. Ratios in Euclidean geometry for example, become algebraic expressions in co-ordinate geometry. Thus, AB:BC becomes $x_2 - x_1 : y_3 - y_2$ for a given right-angled triangle.



2. See the works of F. W. Land, e.g. The Language of Mathematics, for support for such a view.

restrictive demands, identified in Tarski's Semantic Theory of Truth, could have the richness that one associates with ordinary language. Even accepting this difficulty, mathematicians have used the notion of 'mathematics as a language' more as a means to defend themselves against outsiders than to offer a positive perspective. The argument has been that there is an essential 'fluency' to mathematical understanding,[1] comparable to that implied in knowledge of a foreign language. Until one is at the level of appreciating technical terms in this 'language' then one is hardly able to contribute to discussions either in it, or even about it. This leaves the learner somewhat unprotected from the overriding authority of the teacher,

Certainly there are uses of the word 'language' which are specialised. One may talk of 'computer languages', 'the language of music' and in the previous chapter mention was made of 'formal languages' but none of these can be identified with the concept of an 'ordinary language'. Some features are to be found in common and one may take into these artificial languages other words from ordinary language and modify their usage. One thinks of the use of 'meaning' and 'grammar' for example.[2] To be told to teach mathematics as if it were a language does not seem to give very much of a guide to a young teacher. There is still the further question, in what context are the pupils to speak the language. In other words, one may still wonder whether or not the pupils are to speak this language when uncovering a perfect proof, or discovering a new relationship, or devising a new form of Battleships, or producing a model of the different effects of a set of chlorines in the school swimming pool. The differing approaches are not erased by the idea that 'mathematics is a language', unless one associates 'language' purely with 'symbolic systems' and 'games',

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1. The notions of 'fluency' and 'understanding' are discussed further in part 3.

2. Points about 'language' and 'mathematics' recur in part 3.

which is not the sense discussed here. Teachers do seem concerned with mathematics as more than the symbolic manipulation of strokes, although they may well teach mathematics according to this philosophical movement sometimes.

It should be clear that the suggestion is that a particular perspective may be used by different teachers for different reasons. Only some teachers will tend to be technologically orientated for philosophical reasons. Others will be so because material is thought by them to increase the chances of their students getting employment. The most that one could claim is that such a teacher is an 'implicit' pragmatist. Similarly the popularity in the last fifteen years, of viewing mathematics as a formal system or set of systems whose foundations must be made clear to the pupils, does not indicate that all teachers became Formalists. One might suggest that many simply thought all the examiners were Formalists.

CONCLUSION

Given this debate the young teacher is wondering whether there are any hard and fast rules to the teaching of mathematics. This chapter is an introduction to the area, and as such has attempted to isolate some questions that are considered in subsequent chapters of this part of the thesis. The young teacher has employed a range of approaches, due to the different influences on him, but without arriving at a coherent framework for his teaching of mathematics. Therefore it would be natural for him to want to have a systematic outline of the various approaches, which involves consideration of the following questions:

1) What distinguishes perspective X from all the others?

2) Must one logically have come to appreciate one perspective before, as a pupil, one can appreciate some other perspective?

3) Does one find a common logical sequence to what is taught, no matter through which approach it is taught?

These questions may not be fully answered by the end of this part 2 of the thesis, but they do provide a focus for what is to follow. The answers that are achieved provide the framework in which the knowledge and understanding to be expected of any teacher of mathematics in secondary schools today, can be identified. In the third part of the thesis, questions about the training of mathematics educators can then come to the fore.

C H A P T E R 7

MATHEMATICS TEACHING - ART PERSPECTIVE

INTRODUCTION

When the young teacher, met in the previous chapter, considers mathematics teaching as analogous to art teaching, certain objectives come to mind. In his sixth form class (see p.100 above) he expects his students to,

1) 'value mathematics "in all its beauty" as he has done', as they learn to

2a) 'appreciate the elegance and rigour of the proofs',

b) '"create" proofs of their own', and

c) 'compare...proofs...to see which are "better" than which with respect to parsinomy, elegance, et al.'.

He believes that the result of meeting objectives 2a) to 2c) is to progress to 1). It is hoped that in achieving 2) students will have the opportunity to appreciate the historical dimension in which certain proofs came to be written. E. T. Bell in his famous Men of Mathematics speaks of co-ordinate geometry as, 'of the highest order of excellence, marked by the sensuous simplicity of the half-dozen or so great contributions of all time to mathematics. Descartes remade geometry and made modern geometry possible.' (Men of Mathematics, p. 71).

The eloquence of Bell's book does not show that aesthetic language is simply transferable to a mathematical context. 'Words' by themselves do not tell one very much. One must look behind to the theory that underpins them. The language of aesthetics is found to rest on at least as many possible foundations as were considered for mathematics itself, in part 1 of the thesis. To make progress some theoretical frameworks must be sketched in. This will be the objective of the next section of this chapter. With these drawn, the remaining sections will concentrate on the possible links between aesthetics[1] and mathematics teaching. It will be shown that aesthetic theories and the mathematical movements of the first part of the thesis have some features in common.[2] This chapter will try to consider the validity of presenting mathematics to children as an area in which aesthetics is of singular importance.

AESTHETIC THEORIES

In the first part of the thesis Platonic realism was a recurrent phrase. Plato's view of art and aesthetic criticism is compatible with such a realism. The analysis will start with this theory and continue with those roughly parallel to the Formalist and Intuitionist movements of part 1. Mathematical Formalists and Aesthetic Formalists have a common focus on 'ordered structures'. Intuitionists and Expressionists have their common focus on man's inner states, his ways of reasoning and self-expression.

1) Representational Theory

While there have been developments of this theory over the last two thousand years, Plato's presentation of the theory adequately exemplifies the main characteristics of the theory as it comes into contact with mathematics, in this thesis.[3] Plato's view of art echoes

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1. 'Aesthetics' is to be taken here to cover 'philosophy of art', 'aesthetic experiences' and 'aesthetic criticism'. It should be noted also that an object may warrant aesthetic criticism, without its being a work of art.

2. It was noted on p.13 that one could hold a realist view of the world for example, but an anti-realist view of mathematics. Thus a realist view of art may be linked to an anti-realist view of mathematics when one considers the teaching of mathematics.

3. Developments in this theory, as found particularly in Gombrich, indicate a loosening in the absolute nature of the representation, and an acceptance of the Kantian view that experiences have meaning only if filtered through some categorial framework. Thus, in contemporary forms of Representationalism, the artist presents an interpretation of his experiences necessarily, and neither he nor any critic can have the 'pure view' that Plato suggests.

the beliefs of the Pythagoreans nearly two hundred years before. Plato had taken their commitment to the perfection of mathematics as a paradigm of 'knowledge'. The Pythagoreans had seen mathematical truths as the key to all knowledge, including questions about 'the beautiful' and 'the good'.

In Plato's works, his Forms[1] also draw together what is morally right, what is aesthetically beautiful and what is philosophically known. Just as the true proposition '2 + 2 = 4' corresponds to the form of 'truth' in a Platonic reality, the aesthetic claim that the Mona Lisa is beautiful must relate to the form of beauty. Furthermore, what is beautiful must be also morally good, for they correspond to one form.

Thus Plato divides the Arts into those that are supportive of the moral values of the state, and those that show weaknesses in the state's heroes. The artist has a moral duty to represent what is morally good, and to omit the suggestion that those who are morally good can falter. Art is seen as part of this moral education. Thus art is bad, both ethically and aesthetically, if it inhibits this propagation. In such a theory of art, one does not rely on 'Intuition' or 'innate ideas' to make aesthetic criticisms, but the Guardians of the Republic 'know' what is 'good' and 'beautiful'. Just as 2 + 2 is equal to 4 because of something external to man, according to Platonic realism aesthetic judgments hold as eternally,[2] through their relationship to something external. Thus representation provides the definition of what is 'true', 'good' or 'beautiful', but the criterion by which such things are identified is the application of the highest kind of reasoning as possessed only by the

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1. 'The forms' are introduced on pp.14-8 above. They exist in a transcendental reality. That is where an object in its true self is to be found. Thus 'real knowledge...(is possessed by)...those who contemplate things as they are in themselves...and as they exist ever permanent and immutable...' (The Republic, para. 479/480).

2. An identification of the use to be given to 'a priori' in this thesis was made on p. 76 above. It is based on Kant's interpretation, that such knowledge is gained without empirical investigation.

Guardians. One deduces from given hypotheses that '2 + 2 = 4' and by dialectic reasoning that the Mona Lisa is beautiful. Furthermore, one may recognise that something is 'elegant' without knowing whether it is beautiful or not, just as one may recognise that all circles have something in common besides their shape, without being able to prove a link between diameters and circumferences. In each of these cases, one recognises an intermediary form, without recognising the one ultimate form in which all knowledge shares. One can categorise things as 'elegant' or as 'circles' without knowing any 'true, beautiful' proofs about 'elegant circles'.[1] The logic of Plato's position is that nothing can be both true and have an inconsistency or beautiful and possess a negative aesthetic attribute, like inelegance, for the ultimate form is perfect.

2) <u>Formalist Theory</u>

This theory originated with Fry and Bell[2] was vigorously reformulated by Suzanne Langer,[3] over the past thirty years. These thinkers are drawn together in the belief that there is something unique which all aesthetic objects possess. Art does express things but not like a language. Art is not to be criticised for its representational or semantic structure, but for its 'form' alone. Bell's position claims a very strong autonomy for artistic expression. He argues an aesthetic experience is only to be had in considering an object as an end in itself, free from human interests. In such circumstances one becomes aware of the 'formal significance' of the object, and this is the experience of 'aesthetic emotion'.[4]

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1. This is the position explained by Plato in 'The Divided Line' and 'The Cave' illustrations of <u>Republic</u>, vi and vii.

2. Reliance here is generally on Bell alone, particularly his book <u>Art</u>.

3. Langer's books include <u>Feeling and Form</u> and <u>Philosophy in a New Key</u>. In the latter book she refers to Kant's ideas being reflected in her work.

4. See <u>Art</u>, pp. 6-7, where Bell introduces aesthetic emotion as 'the essential quality...that distinguishes works of art from all other classes of objects'. Towards the end of the book one reads this broader conception, that 'By form the vague, uneasy, and unearthly emotions are transmuted into something definite, logical, and above the earth.' (<u>ibid</u>., p. 283).

The outsider seems inevitably to find all versions of the Formalist theory essentially tautological. There is no public way of coming to identify 'form', or aesthetic experience. Similarly one has no means of refuting the theory for any counter-example may be simply identified as not being a formal feature but one tainted by human interests. There are indications in Bell's writings of the kinds of features that he and others are most concerned to study. Bell concentrates on what he sees as the organising features of a work of art. He focuses on line, mass and colour and excludes consideration of the meaning of the work, or the artist's intention to communicate a message. Bell and other Formalists represent a group trying to defend the rise of abstract art forms in the first three decades of this century. Bell saw literature as so inherently concerned with 'meaning' that it is classified as 'impure art'. One might suggest that this recalls everyday distinctions used by scientists, mathematicians and educators to separate pure from applied science or mathematics. Certainly the strict formalist mathematicians seem to flee to a comparable separating of themselves from all questions of interpretation. Bell similarly insists that one cuts oneself off from 'the significance of life' in making aesthetic judgments. He goes on, 'For, to appreciate a work of art we need bring with us nothing but a sense of form and colour and a knowledge of three-dimensional space.' (<u>Art</u>, p. 25). Bell later stresses that an artist is someone who has seen 'objects as pure forms. We see them as ends in themselves, ... Who has not, once at least in his life, had a sudden vision of landscape as pure form? For once, instead of seeing it as fields and cottages, he has felt it as lines and colours.' (<u>ibid</u>., p. 53). Furthermore, '...Be they artists or lovers of art, mystics or mathematicians, those who achieve ecstasy are those who have freed themselves from the arrogance of humanity.' (<u>ibid</u>., p.70).

In modifying Formalism, writers like Langer have continued to accept the primacy of 'significant form' but have admitted that 'meaning' can play a part if the influence is recognised and controllable. This is similar to Hilbert's initial constraint of mathematical Formalism to 'formally consistent systems'.[1] Langer attempted to take the pure position into music but found herself having to give some 'formulation and representation of emotions, moods, mental tensions and resolutions - a "logical picture" of sentient, responsive life.' (This is brought out particularly in her Philosophy in a New Key, chapter 8). She draws the key features of Formalism together with an acceptance that art is expressive in a sense that goes beyond the isolated state of art that satisfied Bell. Langer accepts that success in art includes conveying formal features that are common to the arts and to human feelings. In other words, there are features by which one identifies music as 'sad', because there is some association with what one senses in finding some person 'sad'. While Langer hooks art on to the world, she does not explain how the outsider comes to recognise the commonality between feelings in art and in life. At this point, one is again held at bay, as one is with Bell's original Formalism, by the sui generis nature of art and its symbolism. Appreciation of the correctness of the theory can only come once one has got inside aesthetic appreciation in the formal sense. The circularity may not be vicious, but it certainly seems to be protective.

It was indicated above, in the quotation from p. 70 of Art, that Clive Bell draws connections between art and mathematics. He certainly considered the mathematician to have a similar sensibility to that felt by artists and their critics. He says on p. 25 of Art, 'The pure mathematician rapt in his studies knows a state of mind which I take to be similar, if not identical....I wonder, sometimes, whether the appreciators

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1. On p. 32 above, it was noted that Hilbert, unlike the later Formalists, had a place for meaning in mathematics.

of art and of mathematical solutions are not even more closely allied.' One may feel justified therefore, in saying that the two kinds of Formalism, one artistic and one mathematical, differ, as one 'is the study of significant form and the other is the study of formal systems'.[1] The Formalist theory of art reminds one very strongly of 'a secret society' and this feature of strict mathematical formalism is one that will be returned to, both at the end of this part of the thesis and throughout the next and final part.

Certainly the Formalist requires aesthetic appreciation to occur without reference to other considerations, like ethical or social questions. Aesthetic significance may be noted whenever aesthetic emotion is provoked, no matter what provokes it. However, the applying of aesthetic terminology is to be done with detachment and the linguistic expression is to be recognised as meaningful only within the aesthetic domain.

3) Expression Theory

Croce[2] and Collingwood[3] present works of art as internal constructs, expressions of the self. The physical representation of art is necessary to make it possible for the viewer to get into the artist's feelings, whereby the work of art can be fully grasped.[4] This theory has strong

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1. Even if Hardy saw that 'beauty is the first test' for the mathematician as well as the artist, the link of mathematics and art could be considered intimate but not inclusive. This limitation is well expressed in the article, 'Mathematics and Art', Graduate Training of Mathematics Teachers (pp. 23-30), where K. O. Mays says, 'Patterns in either field may illustrate, explain or inspire work in the other' (p. 29), but he clearly stops short of claiming that mathematics and art can be one.

2. The founder of the theory.

3. The Principles of Art contains his theory of art.

4. R. K. Elliott's article 'Aesthetic Theory and the Experience of Art' in Aesthetics, includes a broadened theory that includes 'arousal of emotion' as well as the sensing of emotion, as a feature of 'aesthetic significance'. Even within the distinction between arousal and sensing of emotions is the further distinction between emotional responses that clearly lie within the control of the artist and those that arise independently of the artist's intentions. Clearly, the latter position is that most commonly implied when commenting on mathematics aesthetically.

similarities to Intuitionism as described in part 1, Chapter 3 above. This is reinforced in Collingwood by comments like, 'The artistic activity does not "use" a "ready-made language", it "creates" language as it goes along.' (Principles of Art, p. 275).

Croce accepted two sides to the experience of a work of art. One does not only try to experience the work of art as an object, to be made part of one's own experiences 'from within', but one must recognise the value also in seeing the work of art as an external object, 'from without'. This is comparable to some Intuitionists in mathematics treating Intuitionistic logic as a valid part of mathematics, and not just the instrument by which one gets to mathematics itself. Croce was concerned throughout his philosophy to eliminate dualism[1] and he saw that the viewing of art as separately a 'within part' and a 'without part' as a dualism. To Croce, an artistic experience only commences if both features were present.

The theory has a simple notion of aesthetic achievement by an artist. It is well expressed in this unspecified quotation from Coleridge in Collingwood: 'we know a man for a poet by the fact that he makes us poets. We know that he is expressing his emotions by the fact that he is enabling us to express ours.' (Principles of Art, p. 118).

Underlying this theory is the egalitarian principle that has significant educational implications, that everyone has 'imagination' and through one's imagination anyone can come to express the emotions one possesses. Everyone has these ordinary, real life emotions, but only artists are able to put them into a form that others can sense. In so reacting, one then learns something new about oneself, further knowledge

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1. 'Dualism' can be taken here to refer to any theory that proposes that a single identifiable object consists of components belonging to more than one ontological realm. The usual thesis given is Descartes' view of man as both 'physical' and 'spiritual', to exemplify such a position.

about one's own emotions. This is not a theory of causal responses. The critic may well have to work hard to indicate his knowledge of a book, say, before others come to sense what it expresses. This reflects accurately the struggle interpreters of great mathematicians have in providing a public form of their achievements.[1]

Thus Expressionism may help to clarify an approach to mathematics teaching most readily considered by Intuitionists. However, Expressionism cannot be directly applied to mathematics as identified by Intuitionists or any other philosophical movement discussed in this thesis, for mathematical objects are not connected <u>per se</u>, with the evoking of human emotions. The validity of a pedagogic link is based on the idea that a pupil may the more readily appreciate what is involved in understanding a mathematical proof, if he compares this experience with getting to know a work of art.

There is little to suggest that aesthetic terminology as conceived by Expressionism is readily transferable to mathematics, unless mathematics is seen to be connected with the expression of emotions. To the expressionist, 'elegance' is not <u>per se</u> an aesthetic term but may be used within an aesthetic judgment, if the elegance of the leading lady's performance say, evoked awareness of deep respect. In contrast, the mathematician can judge one proof as more elegant than another, and therefore consider one superior to another, even though neither proof

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1. This is typified by Heyting's remark, 'As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that (a) formal system expresses correctly our mathematical thoughts.' (Quoted by Dummett, 'Wittgenstein's Philosophy of Mathematics' in Dummett, p. 184). Wittgenstein further clarifies this point in noting that the debate is not resolved by some psychological explanation but by mutual agreement (<u>Lectures and Conversations on Aesthetics</u>, p. 21).

evokes any obvious awareness[1] of pleasure or of any other emotion.

Thus each of the three theories discussed in this section have differing notions of aesthetic criticism. The Platonic Representationalist relies on knowledge of universal aesthetic Forms which correct reasoning identifies. The Formalist focuses on the observation of formal features, detached from questions of content, and these features are seen to provoke aesthetic emotion to differing degrees, according to how they are combined. Finally, the Expressionist requires the critic, in his evaluation, to try to take cognisance of what the artist felt in creating the work being considered, in addition to the more immediate or 'outside' appreciation of the work.

AESTHETIC THEORIES AND MATHEMATICS

There are two possible foci for any philosophical account of art. There is the work of art itself, and there is the criticism of it. In this section, the possibilities and the limitations of aesthetic criticism as a meaningful part of mathematical study will be considered. In the next section, creativity and mathematics will be the centre of discussion.

The simplest objective isolated on p. 109 was 2(c).[2] It is easier to say whether one prefers strawberry or vanilla ice cream than to say in isolation, 'what is it that makes strawberry ice cream so delightful?' The first is a concrete comparison while the second is a comparison with all abstract possibilities. Underlying such comparisons is the assumption that it makes sense to put mathematical proofs say, in an aesthetic order of merit. In the following discussion awareness will be shown of

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1. 'Awareness' is a better term than 'feelings' which might have been expected, for one could follow Expressionism without claiming that the aesthetic object makes me joyful, but only that it makes me aware that it conveys a sense of joy. The critical point for any Expressionist as he arises in the present discussion is that the object contemplated must logically convey emotion to warrant any aesthetic description.

2. 2(c) 'compare...proofs...to see which are "better" than which with respect to parsimony, elegance, *et al*.'
   2(a) 'appreciate the elegance and rigour of the proofs'.

three questions that can be considered to have an increasing order of particularity:

1) What is involved in valuing something aesthetically?

2) What is involved in valuing one thing more than another, aesthetically?

3) What is involved in valuing one part of mathematics more than another, on the basis of aesthetic criteria?

These questions can be seen to reflect the concern of the young teacher, not simply in achieving objective 2(c), but also 2(a); and both in line with the overall objective, to 'value mathematics "in all its beauty" as he has done'.

(i) Platonic Representationalism has a clear criterion for evaluating works of art. The works are as good as their imitation of the forms. According to Plato, one can be sure in one's judgment of the aesthetic merit of a work of art, as it succeeds in reflecting or not the form of 'beauty'. At least in the Republic, Plato expresses a similar view of mathematical proofs. If they are correct then they will reflect the form of 'truth'. Given Plato's view that there is only a single ultimate form, whether it is called 'good', 'beauty' or 'truth', then there is a sense in which mathematical proofs and works of art, if approved, reflect one form. Even if one accepts that what is true is also beautiful, this does not provide criteria for one to go on and put those beautiful objects in sequence. One could not order parts of mathematics by means of this theory, but only divide the objects into two classes; those that represent the truth, and those that do not.

Platonic Representationalism provides a set of classes with the choice of membership or non-membership, while the language of aesthetics as present in ordinary usage is semantically open. One object can be more 'elegant' than another, and so can one mathematical proof be. The term 'elegant' as normally used indicates a quality of which something can have a relatively greater or less a measure. Plato's model of

representation omits the possibility of this more common usage, and in so doing makes the ordering of aesthetic objects logically impossible.[1]
(ii) For the Formalist aesthetician, aesthetic appreciation occurs in a semantically open set of symbols, unlike that of mathematics which is closed. Any attempt to have semantic rules would be frustrated because they would be more honoured in the breach than in their observance. Bell provides no clear clues for progress but Langer identifies 'formal features', isolatable syntax. However, these too have no definite rules to govern them but one may expect their 'presence' in all works of art. This provides some basis for discussion, but it does not provide even a perfect being, with some infallible method to order a set of dances, say. The Formalist's concern for structure does not extend to public grading criteria.

Formalism has an admitted dominance in mathematics, and one might look for a similar claim in aesthetics. However, such a search would take this thesis too far away from its central purposes. In mathematics, one proof relies on a considerable number of definitions, followed by short proofs, while another relies on just two handfuls of postulates but the proof itself is long and intricate (compare the trigonometric with the Euclidean proof of the converse of Pythagoras' theorem, that if $c^2 = a^2 + b^2$ then the triangle is right-angled). The form of each proof may be quite distinct, both on immediate impression and even after careful study. The one proof is simple once begun, while in the other one meets a new construction or condition on every other line. It feels more like a conjuring trick than a mathematical proof but once completed, mystery is turned to clarity. In exactly the same way one may distinguish

1. Logically impossible that is, if aesthetic qualities are taken to be of equal importance in participating in the ultimate Form of 'beauty'. While discussion of this point is possible, it is not critical to the overall argument of this thesis.

two novels. In one say, (Zola's Nana) all the characters are presented early, while in another (say, Uris' QBVII) characters are held back to give twists or landmines as one travels through it.

Certainly the Formalist[1] mathematician would seem to have the likelihood of a happier relationship with an aesthetic theory that concentrates on universal 'structures', rather than one that rests on a conventionally agreed language. Thus there is a strong force linking mathematical Formalism to aesthetic Formalism. Yet part of the force's strength may be seen to highlight a weakness in the aesthetic theory. While the aesthetic Formalists, like their mathematical counterparts, identify universal structures, the aestheticians have yet to agree on criteria by which to identify the structures. The struggle over 'undecidability' in mathematics is as nothing when compared with the disputes among aesthetic Formalists, particularly as the mathematical Formalists have wide agreement over the treatment of finite systems. The aesthetic Formalist may not aid his mathematical counterpart as yet to order his mathematics, but it can be noted that the two movements have similar underlying premises and could serve each other in the future.

(iii) The fundamental problem for the Expressionists, as well as for the Formalists, is that in inheriting Kantian origins, they have inherited

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1. This is not to exclude the use of aesthetic Formalist criteria to show appreciation of mathematics, by people who are not themselves mathematical Formalists. The critical presumption is that parts of mathematics can provoke aesthetic emotion. Aesthetic emotion 'is more intense and focused than the ordinary emotion we experience in our daily lives. Mathematicians have said the same of their experience of pure mathematical relations.' (J. Wechler's Introduction to On Aesthetics in Science, p. 6). She certainly seems to assume that in some sense, these mathematicians could admit such an experience as being one of aesthetic emotion, and not just analogous to it. If aesthetic emotion provides a formal framework for all emotions then it is not unreasonable to find a formal subject like mathematics evoking this kind of emotion, even though mathematics per se does not evoke ordinary, real-life emotions.

'aesthetic intuition' without his 'aesthetic conceptualisation'.[1] Aesthetic judgments are precisely those that are not determined by knowledge (covered in Kant's Critique of Pure Reason), and also those not covered by universalisability (Critique of Practical Reason). These judgments are ultimately subjective. Ordering is not ruled out but it cannot be finally objectified either.

Elliott[2] accepts this but tries to explain how an Expressionist could show preference. He indicates what it must involve. A particular work of art may have more power in drawing one into it, if it provides sufficient clues so that one can view it from within. Elliott sees this as magical; the magic needed to see clues clearly. This is not objective, for one person or many people may be blind to the clues. The fault may lie more with the viewer than the artist, but for the Expressionist these clues are the only route to aesthetic criticism.

Scruton[3] seems to make explicit one more feature of Expressionist criticism that is relevant to an educational discussion. In getting inside a work of art, the viewer is going to gain from the experience. The viewer gains in educational development. Scruton argues that a

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1. Langer accuses the Expressionists of conflating 'form' and 'intuition' and that the result is 'confusion' in their theory (Feeling and Form, pp. 375-6). Certainly both the Expressionists and the Formalists have taken on board Kantian ideas. In the Critique of Judgment Kant writes, 'I need not know its material purposiveness (the purpose), but its mere form pleases by itself in the act of judging it without any knowledge of purpose.' (Para. 48). Later, para. 59, 'The beautiful pleases immediately (but only in reflective intuition, not, like morality in its concept).' The two theories have inheritances in 'mere form' and 'reflective intuition' which is employed both in criticism and creation but there is room for developments in these theories as Kant had considered both these dimensions of 'sensibility' and their control under 'the laws of logic' even earlier in the Introduction to Logic, pp. 24ff. There is the basis of an argument that Wittgenstein, as well as Langer, considered Kant's fuller thesis seriously, but there is not room here to elaborate on this. However, signs are found in Lectures and Conversations on Aesthetics, particularly in Lecture IV, pp. 28-36, but also in Lecture I.
2. Op. cit., pp. 154ff. (Footnote 1, p. 115 above).
3. Art and Imagination, pp. 246-9.

necessarily worthwhile experience like that achieved in the making of an aesthetic criticism, is only ultimately justified as such, in terms of moral values. Scruton's fuller thesis, which will not be discussed here, is not just that 'worthwhile' carries moral approval, but that aesthetic judgments and moral judgments are one. 'Ethics and aesthetics are one.'[1] This would seem to have been Plato's position too. The resultant criterion for ordering works of art is therefore their worthwhileness, as ethically evaluated.

Intuitionism as a study of mathematics cannot be seen as synonymous with Expressionism as a study of art, but it can be seen to have parallel formal features. The Expressionist uses public features to guide someone into the human feelings that the work of art expresses. The intuitionist uses the public symbols of intuitionistic logic to guide someone into those features of human reasoning that are used to construct mathematics. Understanding mathematical proofs can be taken as analogous to appreciating works of art. A pupil achieves the understanding through the re-establishing of a proof, that Descartes derived, or calculating a square-root, as Pythagoras once did. The pupil attempts to share in the experiences and thereby to come to have a feel for what were, at one time, leaps of knowledge. It would be surprising if one went on to claim that the mathematical proof could be seen to express human emotions. People may accept that what is recognised in a proof as 'the self-evidence of wholeness', 'richness of stature', 'amusing procedures' or 'chilling simplicity',[2] are aesthetic qualities, but not that these are necessarily open to redescription in terms of human emotions.

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1. Scruton's concluding remark on p. 249 of <u>Art and Imagination</u>, culminates his argument that public standards of critical and ethical judgment are interdependent.

2. This would demand an extensive study of the philosophy of the emotions to justify, but what does seem to indicate itself is the possibility that formal features like 'simplicity' can be associated with human feelings in appropriate circumstances. Furthermore, the basis of such an association may well be 'imagination' as Scruton identifies it in <u>Art and Imagination</u>, chapter 9.

CONCLUSION

An important feature of the discussion in the last paragraph has been that whatever one identifies as aesthetic or other qualities, the comments could not be made intelligently if one had not sufficient relevant understanding of the subject matter of mathematics. The critic must have sufficient understanding of mathematics to 're-establish the proof' and experience 'the self-evidence' or that 'the procedure employed at some point really takes one by surprise', or whatever. This is not highlighting a point relevant only to Expressionism, for the Formalists and Representationalists also require the critic to have some understanding of the area to be aesthetically appreciated. As a matter of fact, once this condition is satisfied, one may aesthetically evaluate any piece of mathematics without fear of contradiction, whatever one's philosophy of mathematics. However the kind of knowledge required[1] for such criticism may differ according to the aesthetic theory:

(i) The Platonist requires understanding of what conditions are identifiable pre-requisites for the proof, and then for actually completing the proof.

(ii) The Formalist would be similarly concerned with the principles of manipulation required.

(iii) The Expressionist requires an attachment to human emotions that is not generally recognised by mathematicians and so any link with Expressionism will be no more than in an analogy. As such, the Intuitionist would have most in common with Expressionism, for both these movements treat physical constructs as signs of a fuller reality. The Intuitionist mathematics teacher has a view of his subject as possessing

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1. It is not within the limits of this thesis to identify what is required to understand a given proof, but further points could be made as to whether this kind of analysis is really a branch of psychology or not, as seems to be the argument of R. R. Skemp (The Psychology of Learning Mathematics) among others.

an 'inner' and 'outer' component that parallels the Expressionist's view of art, and so any analogy of mathematics as aesthetic can be more readily drawn out of this connection, than between any other philosophical view of mathematics, that links 'inner' and 'outer' components contingently, and Expressionism.

Thus Expressionism can provide a basis for the criticism of mathematical proofs aesthetically, only in a very limited sense. Platonic Representationalism provides the framework for such criticism but no basis for ordering the identified proofs further, in terms of what are commonly considered aesthetic criteria. Only aesthetic Formalism seems to give a framework that could be applied to mathematics, without excessive discomfort. This application might well be facilitated if the mathematician, through his philosophy of mathematics, is keen to pick out the formal features of mathematics as of greatest importance, as the mathematical Formalist would do. Naturally, aesthetic Formalism could cohere with other philosophies of mathematics, but not readily. The Logicist would have no logical reason for objecting to such an approach, but the Intuitionist could argue that aesthetic Formalism looks for universal structures in public objects and in so doing might fail to look at mathematics per se. It would be looking at the arbitrarily chosen frameworks that facilitate the publicity of mathematics, and not mathematics itself. The Hypothesisor's objection is the more general one that mathematics is not the kind of subject to which it is appropriate to claim that the subject itself will benefit by aesthetic analysis. The Hypothesisor rejects the idea that mathematicians can turn to aesthetics to order mathematical proofs for their 'applicability', and aesthetic Formalism is particularly inappropriate for it lays great emphasis on its lack of concern in criticism on interpretation. Aesthetic terminology[1] may be still appended to mathematical proofs as a matter of

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1. Wittgenstein argues that there is a problem of logical impenetrability in a discussion of art. One cannot translate art into ordinary language. The only really successful way of commenting on a painting is 'again to paint', for 'you can't at all transmit the impression by words' (Lectures and Conversations on Aesthetics, p. 39). Wittgenstein indicates the difficulties involved in moving from one identifiable 'form of life' to another. Success in communication within and across the resulting language-games must be seen, according to Wittgenstein, ibid., pp. 37-40, in relation to a given culture. Implicit in the move away from Plato, is the dislike of both his epistemological and ontological realism, discussed in Chapter 1 above. The ontological realism provides a difficult problem that Wittgenstein solves in the area of mathematics, discussed on pp. 46 and 47 above, by treating mathematics as 'rules' and not 'objects' at all. Croce, in the area of art, claims that art belongs to neither a real nor an ideal world. This is part of the justification for his remaining free of the dualism, mentioned on p.116 above, but Croce does not give any clearer solution to this ontological problem. One solution has been mentioned above, p. 65, that one turns like Popper to 'the world of all that man may think up'. This includes for Popper, both the aesthetic and the mathematical realms. The validity of the option cannot be fully discussed here, but that the problem has lifted its head again, is surely worth recording. Conventions have to play a part in both artistic production and also in any aesthetic criticism. The artist is constrained by the conventions of his medium and of his art-form, and the critic is constrained by his having to move from the thoughts of one language-game to those of another, and this can only occur, even ambiguously, if there are agreed conventions. This position is comparable to that felt by a physicist today, who wishes to describe his experiences in experimenting with 'x-ray scatter in treatment', for example. In x-raying patients different kinds of nozzles can be put on the machine, and the resulting link between exactness of x-ray and scatter of ray can be expressed mathematically. The only language that he can use is mathematics, and there is no translation into the language of medical science that could adequately convey the meaning, beyond 'doing it this way is more dangerous for the patient than that way is!'. If one accepts that art is *sui generis* then commenting upon art must consist of using conventionally accepted terms, that try to pick out as well as possible the features of the 'form of life'. The success of the resulting communication partly depends on those involved being aware of the constraints under which it occurs. Thus the enlarging of the language of aesthetic criticism to cover mathematics will involve further agreements to those presumably already acknowledged for the Arts, as previously identified. Thus, it may be agreed that 'parsimony' can be used to indicate the finesses of a trigonometric proof of Pythagoras' theorem, as well as Hamlet's 'To be or not to be' soliloquy. Furthermore, any ordering on the basis of aesthetic criteria would also be a matter of convention. Mathematicians might agree that 'elegance' is to carry more weight than 'parsimony', for example, and so all pupils will continue to study Euclidean as well as trigonometric proofs. However, there is no point in pressing the idea that mathematics is itself an art-form, for no philosophy of mathematics discussed in this thesis presents

[Contd. overleaf

fact, but it would seem to be the mathematical Formalist who is most likely to see the linking of aesthetic criteria to mathematics, as potentially beneficial to the development of mathematics.

BECOMING CREATIVE

In the above discussion one objective was not met.[1] This may be reintroduced as a question, 'How will pupils create proofs of their own?' 'Creativity' in recent years has been given much importance in education.[2] Some points may be usefully identified now. It will be assumed that 'creative' is a complex term and two concepts will be taken as identifiable. These concepts can be called 'hard creativity' and 'soft creativity'.[3]

Hard creativity entails the bringing of something into being. The ultimate example of this is God's creation of the universe out of nothing, but poets and composers may claim similar successes, though from more concrete beginnings. In this sense, being creative indicates originality - the creator has extended the world - a work of art is creative by adding

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Fn. 1, p. 126, contd.
the communication of aesthetic experiences as the core feature of mathematics, rather than the deriving of formal proofs. The major problem of identifying art with propositional form, and only Representationalism accepts such a link, is that one has no clear idea of what falsity is in such a structure. One possible response to someone like Pring who says, 'What conceivably could be the negation of the Mona Lisa?' (Knowledge and Schooling, p. 45) is nothing forces one to use that kind of a logic with bivalence, in it. This reinforces the impossibility of strong connections with mathematics, for even Intuitionists have a clear idea of what falsity is in mathematics, and this is central to what one means by mathematics. Naturally, aesthetic terminology may still be employed in mathematical contexts and be seen as meaningful according to some agreed conventions, without mathematics being itself an art-form, provided there is some context where the terminology does apply to art-forms, like Shakespeare soliloquies.

1. 2(b) 'Students will be able to "create" proofs of their own.'

2. An article by the author, 'Creativity and Mathematical Thinking' is attached to the thesis.

3. While the author used these terms in a lecture at Worcester College of Higher Education in January 1974, and Ormell uses the distinction in 'Bloom's Taxonomy and the Objectives of Education', Educ. Res., 17, 3-18, 1974, it rests upon R. K. Elliott's two types of creativity as enunciated in 'The Concept of Creativity', and 'Versions of Creativity' in succeeding volumes of Proc. Phil. Educ. Soc., 1971.

a new object to some reality. Furthermore, what is created is not a simple thing like a new sentence, but rather, what is created must have a structure that is partly peculiar to it: a form. The creative person produces a new structured object.[1] Elliott argues further, 'Creativeness is part of the concept of an artist, but it is not part of the concept of a scholar or a scientist, and even resists[2] being predicated of them.' For example, Whittle invented the jet engine, but did not create it. He built on to what body of knowledge was already there. There is obviously a short contingent step between the claim that mathematicians are creative in this 'hard sense' and the further claim that mathematics is an art. Elliott has not argued however that only artists are capable of hard creativity, so it does not follow from having incidents of hard creativity within an area that that area is an art form. It only follows that mathematicians cannot claim simultaneously to be both necessarily scientists and necessarily creative in this hard sense. Generally mathematicians may be content with a soft sense of 'creative' and one may well take it that this is how 'create' is used by the young teacher met in Chapter 6, in the objective 2(b).

Soft creativity covers any innovative act of ingenuity or of imagination. This concept may be limited by requiring the act of innovation to be linked to the solution of a problem. Left unrestricted, the concept may be thought to have a 'soft under-belly', by which Einstein and any John Doe who is following a Nuffield 5-13 Science Course may be considered creative under the self-same heading. This may not be significant, for only Einstein could possibly be considered creative in the 'hard' sense, which is not simply a more demanding concept but nearer to the original

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1. 'Object' is taken broadly to cover the various kinds of ontological commitment met in part 1 of the thesis, including 'a new structured set of rules' as Wittgenstein would propose.

2. 'Resists' is not to be taken as 'cannot logically be'.

use of the word. Einstein does not simply make a discovery for himself, but provides a theory that corrects and betters what has gone before, and furthermore, stimulates other new advancements - 'dynamic innovation'.

Given these two concepts, mathematicians may be at times creative under both senses. One might argue that when Boole produced his algebra of sets, there was nothing of its like before and so it was 'hard creativity'; while Descartes may have invented co-ordinate geometry, for he brought together two existent systems, those of algebra and Euclidean geometry. However one draws lines, there may well have been confusion caused in education by worries about the relationship of objectives linked to 'creativity' and those linked to 'discovery'. Under the concept of 'hard' creativity such objectives are separated logically, while if one has a very soft notion of creativity, then 'creativity' and 'discovery' objectives may be synonymous.

The danger lies in both teacher and taught coming to think that what occurs through 'discovery teaching methods'[1] is 'hard' creativity, when this is generally incompatible with expectations based upon the scientific method. Either the scientific method is of secondary importance to aesthetic development in mathematics or it is accepted that only some quite exceptional person who has revolutionary vision can be called creative in the 'hard' sense.[2]

While creativity in the 'hard' sense is linked to the artist and in schools one may talk generally of children who are writing or painting as being free to be creative in the 'soft' sense, nothing has yet been said

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1. This teaching method and the place of scientific methods in mathematics education is discussed much more fully in Chapter 9 below.

2. This clears away part of the problem about mathematics being approached as an art. To demonstrate that mathematics is an art, one would have to decide upon which theory of art, it is an art. It may be easy to show that a Formalist notion of art links music and mathematics nicely together, but then one has the problem of separating them in order to preserve, if one wishes to, the <u>status quo</u> in which mathematics has five compulsory periods on most school time-tables and music, no more than one.

to indicate that 'creativity' is itself an aesthetic quality. It may be often presumed that it stands alongside 'beauty', but this is surely a misplacement. When Einstein is considered creative in the 'hard' sense, he is still considered a scientist and there need not be any feature of his work which is beautiful or even aesthetically surprising. 'Creativity' may most commonly categorise aesthetic activities, but it does not categorise such activities exclusively. Thus, objective 2(b), that students '"create" proofs of their own' could be considered by any mathematics teacher without his ever considering the more general objective 1), that students 'value mathematics "in all its beauty" as he has done', though not vice versa.

Independent of aesthetic considerations, mathematics is presented by all philosophies of mathematics met in this thesis as an activity that involves consistent reasoning, and by this means, at least in part, mathematics is developed. No philosophy presents mathematics as developing either spontaneously or accidentally. A child who wrote 'pi x r^2' on a piece of paper would be considered as acting under hypnosis or some such state, rather than a mathematical genius, if it became clear that he could make no mathematical use of the symbolism. Similarly, the six year-old who writes '7 + E = 10' for '7 + 3 = 10' has not discovered algebra but has problems of laterality, for example. Only a child who understood what linear equations involved could be 'creatively' producing '7 + E = 10'. As was pointed out on p.100 above, the young teacher will lead his sixth formers through proofs and <u>then</u> leave them to their own devices to 'create', he will not expect 'creativity' to gush spontaneously from ignorance. Nothing written in this chapter or elsewhere[1] should make one believe that the author considers that creativity could arise in such a fashion.

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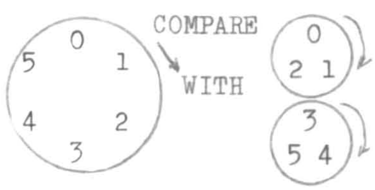
1. In the article, 'Creativity and Mathematical Thinking' which is attached to this thesis, the author elaborates his view of creativity and education.

Practical Interlude. Before ending this chapter, an example of 'creative' work may clarify the discussion. A class of 11 year-olds are introduced to Modulo arithmetic via a discussion of the days of the week. That is,

Sun.,	Mon.,	Tues.,	Wed.,	Thur.,	Fri.,	Sat.
0	1	2	3	4	5	6

Modulo 7 and the days of the week correspond. The class soon produce an addition table and recognise that the same structure underpins both, 'How many days after Thursday is Tuesday?' and '5 + _ = 2'. One of the class is soon most competently dealing with all that the teacher presents. The child is fascinated by the notion of this as 'Clock arithmetic' and sets to work on 'Figure eight arithmetic'. The child is surprised to find that in his new system '+' is not commutative'. '$a + b \neq b + a$'.



While '$2 + 4 = 0$', $4 + 2 = 3$'. The child feels that he has something fantastic here. Nothing short of a 'creation'.

All educators seem to agree that in teaching mathematics,

> We are concerned with the creative side of the child's learning and with minimizing the teacher's interference with this. Every time a teacher insists on his way of doing a piece of mathematics which do not seem to fit, he nibbles away at the pupils' ability to act mathematically. We believe in the value of the child's mathematics.
> (A.T.M. Mathematics in Primary Schools, pp. 4-5)

CONCLUSION

Though not succeeding in answering the questions laid out at the end of the previous chapter, except to a limited extent, nevertheless a compilation of evidence has been produced, together with some suggestions for the young teacher considering this approach.

In this chapter three aesthetic theories have been considered: Platonic Representationalism, Expressionism and Formalism. It was found that this form of Representationalism provides a very limited view of aesthetic criticism which for the mathematician provides no equipment for choosing the more 'elegant' among proofs, other than the tools he already possesses in order to understand what mathematics is and how proofs are derived. Expressionism has the potential equipment for ordering objects aesthetically but it ties aesthetic criticism to the expression of human emotions, and one does not generally associate mathematics with such sensuous appeal. However, the concern with 'inner states' shown by Expressionists does seem to resemble the Intuitionist view of mathematics and may be the source of useful analogies for the mathematics teacher interested in both Expressionism and Intuitionism. Finally, aesthetic Formalism was found more readily to fit some views of mathematics rather than others. The concentration upon formal features, independent of interpretations, parallels well the view of mathematics that mathematical Formalists and Bourbakists[1] would seem most likely to support. Thus the mathematics teacher who is concerned with the structural properties of mathematics may easily find aesthetic Formalism at least a source of motivating analogies through comparisons with music and abstract art, for example. Furthermore, the quality of 'significant form' may well be considered the obvious basis for selecting the more 'elegant' of two proofs, and so on.

A number of points about creativity have also been clarified. It has been indicated that its connection with art is not as an aesthetic quality, but as a criterion of being an artist. This leaves the possible links with mathematics freer and more readily acceptable. However, it is to be remembered that whatever one means by creativity, and two senses are identified here, the teacher cannot characterise it as part of an objective that pupils will suddenly achieve independently of their previous knowledge. Particularly with mathematics one must recognise that while 'creativity' in the 'soft' sense need not be considered a

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1. Both movements seek the development of mathematics by formal procedures, wherever possible. Papert for example, argues that mathematicians can be guided by 'mathematical beauty' as a formal principle in their work ('The Mathematical Unconscious' in Wechsler, <u>loc. cit</u>., pp. 104-19).

rarity it cannot be associated with unprepared spontaneity either.

These conclusions and the evidence behind them will be available when, at the end of this part of the thesis, and in the next, comparisons are made among the different approaches. Even if the link of mathematics to the arts perspective is weakened, it will be seen in the next chapter on the Games perspective, that the hard creativity concept can still be usefully discussed, and certainly the general use of aesthetic terminology by pupils of mathematics will be retained. Pupils will continue to be expected to gain and talk about the excitement[1] that mathematics can give.

1. As Papert admits in his article, even for those who believe in the aesthetic component in mathematics, there has to be the admission that the 'exhilarating power' consists of integrating the aesthetic, the functional and the hedonistic. ('The Mathematical Unconscious', loc. cit., pp. 112-14). Demonstrating that in mathematics at least, the aesthetic is separable from the hedonistic could well be the basis of extensive future study. It is presumed to be a realistic possibility in what is written in this thesis.

CHAPTER 8

MATHEMATICS TEACHING - GAMES PERSPECTIVE

INTRODUCTION

This chapter would progress very little without some indication of the use to be given to the term 'game'. The use to be followed here is to be found in 'game of chess' or 'game of cards' rather than 'game of football', 'he's fair game' or 'love games'.[1] Football, tennis and other human rule-governed activities allow exponents an unlimited number of possible moves at any moment. Thus one cannot write a book on 'all football kick-offs' say, as one can find such books on, 'all two move openings' in chess or 'all leads' in bridge.[2] The view of 'game' to be taken as central in this chapter is one that is tied to the stipulative belief that, as chess is a strictly formalised activity, so is mathematics. Human decisions within such formalised activities are not just <u>in accordance</u> to rules, like obedience to laws in everyday life, but are <u>repeatable</u> instants of what is decided.

Compare 'a pass in football' with 'bishop to rook 6 in chess'. The former activity has been chosen from a non-denumerable set of possible actions,none of which could be exactly defined. The latter activity is chosen from a denumerably finite set of possible actions, each of which must therefore be exactly defined. Thus every move in a game of chess can be repeated, while 'repetition' is only possible in some internally prescribed sense for football. That is, the manager decides to repeat

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1. 'Love game' may be taken as either a tennis or sexual activity.
2. In practice this would be restricted to 'all leads using system of type X', e.g. Acol.

that 'free kick move' in the next match, or the television producer provides a 'video-repeat' of the winning goal. In neither case is the replication exact as it is in chess. Replication is a crucial feature of mathematics in any public form, and so one would expect a connecting of mathematics to 'other' games where physical pieces are not essential.[1] The term 'game' certainly has a range of uses but precision of some kind is possible here because the game-context is quite tightly restricted to those people concerned with mathematics teaching who indicate views on one or more of the criteria to be given now. The criteria themselves are taken with some literary licence, from other sources.[2]

1) There are rules of play.

2) 'Play' is a non-serious activity - it has no extrinsic implications. In particular 'playing' does not involve the making of moral decisions in addition to decisions taken in accordance with the rules of the game. Decisions taken within the game are taken seriously.[3]

3) There is at least one rule whose implementation is required to start the game, e.g. throwing a six in Snakes and Ladders.

4) The terminology and, where appropriate the objects of play are contextually (internally) defined.

5) There is a criterion/criteria by which the end of the game is decided, at least in principle.

6) Participation is voluntary and so generally there is an expectation of enjoyment.

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1. Chess could be meaningfully exhibited by two people using paper or nothing at all. What is required are rules of procedure, not physical objects. This is precisely how some blind players play, never needing to touch anything or have a recapitulation.

2. Stimulus for the criteria comes largely from, <u>The Philosophy of Primary Education</u>, R. F. Dearden, pp. 95ff. and 'Ethical Aspects of Sport and Games', D. Aspin, pp. 58-61, in <u>Proceedings of the Philosophy of Education Soc. of Gt. Brit.</u>, Vol. IX.

3. Such decisions are taken according to logical reasoning generally and to maximise outcomes with regard to the allowable completion of the game. Thus one follows rules logically and consistently.

As a matter of fact, knowledge of all the rules of the game is neither a necessary nor a sufficient condition for playing the game. A necessary condition for playing the game is knowledge of some of the rules. Someone may play chess adequately for many years before discovering the rule of '<u>en passant</u>' (taking of a pawn, by an opposing pawn passing through the third rank to the fourth). Thus it is not necessary to know all the rules of chess even, in order to play it. Conversely one may know all the rules of netball but be disallowed from playing it, as one is a man. Thus a teacher may have taught someone successfully all the rules of a game, but that is not an indication that sufficient conditions have been satisfied for the person to play the game.[1]

It would seem possible now to focus in on the matters to be considered directly in this chapter. The central concern is with mathematics presented to pupils <u>as</u> a game. The use of 'as' hides a number of possibilities which require explicit mention. One may be prepared to employ the occasional game, like Battleships to facilitate learning without wishing to encourage identification of mathematics with games. Even if one were prepared for such identification, there would be a difference between someone who believed that mathematics was one undifferentiatable system and someone who believed mathematics was logically separable between 'counting systems' and 'measuring systems'.[2] The former teaches mathematics as one game, while the other teaches it as two games related through their common use of numbers, say. While another teacher only treats games for motivation, a fourth may believe that some part of mathematics is identifiable with games, but not all of it. Four views seem to present themselves:

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1. There is the example of someone crippled who may umpire at tennis.
2. One may divide mathematics into 'discrete elements' and 'continuous elements'. When one counts, each number is separate from the next but in measuring there is no separation of one length and the next.

a) Teaching mathematics as one game that has sub-games developed from it. It has clear criteria by which it is distinguishable from all other games. Thus, analogously one recognises 'Whist' as a game but there are other games developed from it, 'Solo', 'Contract bridge','Auction bridge', et al.

b) Teaching mathematics as a family of games that are linked by the use of common pieces, numbers, signs for identity, and so on. This is similar to a view of all card games forming a 'family', or Ball Games, or Board Games.

c) Teaching mathematics through games, but not identifying it with games. The approach is justified on psychological or general pedagogic criteria, but not upon philosophical links between the form of mathematics and that of games.

d) Teaching mathematics in part, as identifiable with games.

It is not the intention of the following discussion to find examples of educators claiming that the criteria for a game given on the previous page are sufficient to identify mathematics. The examples of teaching mathematics will be discussed in relation to these criteria. Yet the same or other criteria may make mathematics, not only different from other games, but identifiable within some other category as well, for example, as a component in a common curriculum. Furthermore, it will be argued that one's view of the nature of mathematics may itself restrict the relative emphasis of games' criteria.[1]

Given an analysis of these four views it will be possible then to see if they connect logically to the philosophical movements of the first part of the thesis. The remainder of the chapter will consist of discussing the questions identified at the end of part 2, Chapter 6, and

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1. Criterion 2 which identifies a game as non-serious will be seen to conflict with the notion of mathematics as 'Hypothetical', presented in Chapter 4 above.

it will be seen that the 'games perspective' provides a clear indication of how mathematics and its teaching has been 'isolationist'.[1]

MATHEMATICS AND GAMES - FOUR POINTS OF VIEW

a) The 'one game' view

This view is best characterised through the idea that 'Set Theory' provides a fundamental game in terms of which a number of sub-games may be identified. Once this has occurred the sub-games may seem to the outsider to be independent games at the same level as 'set theory' many of whose features they may have shed. This is the view of the Bourbakists,[2] who represent mathematics teaching in higher education, but one can find this view at the Secondary level also. Learning Mathematics (The Shropshire Mathematics Experiment) begins with an introduction to the language of sets. This language is found in every chapter thereafter, except those that might be called 'History of Mathematics' and 'Applied Mathematics'.[3]

Before considering objections to this view, it seems reasonable to provide the basic framework of an argument to justify the idea that mathematics can be treated as one game. The one game selected is 'set theory'. The emphasis of the argument required is that set theory be shown to be identifiable as a game rather than to be shown that 'every sentence expressible in the notation of pure classical mathematics, whether in arithmetic or the calculus or elsewhere, can be paraphrased into this'.[4] It will be assumed that Quine for example, has good, if not conclusive, reasons for holding to the position just quoted. It is

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1. This is the position that seems to exist in mathematics education which claims a peculiar status for mathematics thinking that everyone requires but only mathematicians can identify. This argument and its rejection will be completed in part 3 of the thesis, after discussion in this chapter and in Chapter 11 below.

2. See footnote 1, p. 53 for an introductory identification of Bourbaki.

3. For example, Book 1, Section 19 involves historical discussion of Galileo and also the use of metric equivalents - 'history' and 'practice' rather than 'pure'.

4. See Quine's discussion in 'Foundations of Mathematics' in The Ways of Paradox. The quotation is from p. 32, ibid.

easy to see that the terminology of set theory is contextually defined for given the use of letters as 'variables' one requires just '( )', '-', 'e', '$\wedge$' and '(x)' to provide all the language of set theory. Thus, Quine gives the example,

$(x) - (y) - (x \text{ e } y \wedge - y \text{ e } x)$ as employing all the terms.

Not synonymously but roughly, this may be read as,

'For all x's, there is a y, such that x is a member of y and y is not a member of x'.

The rules of the game are the criteria of deduction and the game begins when a criterion is applied to a fundamental axiom. The game ends when all possible combinations of the language are derived or shown not to be so derivable. In practice one only plays sub-games in which one attempts to derive a given statement in the language.

For example, given $-(A \wedge -B) =_{\text{df.}} A \dashrightarrow B$ and $-(-A \wedge -B) =_{\text{df.}} (A \vee B)$ and $(A \dashrightarrow C) \dashrightarrow ((C \vee A) \dashrightarrow (C \vee B))$ is an axiom. If $A \dashrightarrow B$ and $B \dashrightarrow C$ are theorems, then prove $A \dashrightarrow C$ is a theorem.

$(B \dashrightarrow C) \dashrightarrow ((A \dashrightarrow B) \dashrightarrow (A \dashrightarrow C))$ by substituting in the axiom, B, C and -A respectively.[1]

$(A \dashrightarrow B) \dashrightarrow (A \dashrightarrow C)$ by modus ponens

$(A \dashrightarrow C)$ by modus ponens.

This indicates the way in which set theory may be seen as a game. In an English lesson a teacher may employ an analogous activity of asking the class to change 'HILL' to 'BASE': HILL, BILL, BALL, BALE, BASE.[2] Some indication has now been given of the way in which mathematicians demonstrate the closed and relatively complete nature of mathematics, based on set theory. These mathematicians have no concern for mathematics

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1. This is based on Bourbaki, Theory of Sets, p. 29. The substitution misses out, $(B \dashrightarrow C) \dashrightarrow (\overline{(-A \vee B)} \dashrightarrow \overline{(-A \vee C)})$ and $A \dashrightarrow B \equiv -A \vee B$.

2. No implication is to be drawn that English teaching is ever anything more than taught through games.

beyond its internal formalisation. Their rejection of 'meaningful interpretation'[1] of mathematics as falling within their area reinforces the line of argument given here, that mathematics is seen both as a game, and as an isolated area.

It is now appropriate to consider objections to the above view. The objections considered have educational implications. They could be put against view b) set out on p. 137 above, too, and so their clarification now will facilitate the discussion of view b) that follows. Finally, in moving from the game to its being taught, one principle is assumed. This is the principle that as a matter of fact it is discernable by practising mathematics educators what a pupil requires in the way of knowledge under criteria 1) to 5) (p. 135 above) before he can be said to be ready to play a given sub-game.[2]

Objection 1. Criticism of a game is possible without sufficient knowledge to play it, but this is not possible for mathematics so mathematics cannot be a game. For example, the only condition of 'Fox-hunting' that one may know is that the game is completed when the fox is dead. This knowledge is not sufficient for anyone to play the game but may be held sufficient for one to criticise the game. No such possibility could occur in mathematics. It is not a game in that sense.

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1. It is important to remember that the Bourbaki programme is not a philosophical programme but simply a way of making 'men's hearts beat more easily', 'to provide a solid foundation for the whole body of modern mathematics' (Theory of Sets, p. v). It holds neither to the view that mathematics is one interpreted axiomatisation as Russell and Hilbert believed, nor to the view that it is a collection of formal systems, some more and some less consistent, as Curry and his followers held.

2. This is the condition already identified for aesthetic criticism in the artistic approach, on p. 124 above. N.B. 'Sub-game' has two uses. On the one hand one might call the whole of arithmetic, a sub-game of set theory and on the other, any sub-routine of either may itself be called a sub-game.

Reply 1. This may seem a trivial objection but the question of 'privacy'[1] is central to arguments about mathematics as a game. The matter will not be fully clarified here but will have to be returned to in part 3 of the thesis where the nettle will be squarely grasped. Two types of reply are possible,

A) One may criticise mathematics without knowledge of more than one condition. Someone may learn that there are 'right answers' (proofs) in mathematics and object that such a game is harmful to children who play it. The important point is that criticism by someone lacking in knowledge of the game in itself can only be about the status of the game as a whole. Should fox-hunting be banned OR Should mathematics be taught in school? It cannot be criticism of rules within the game for the 'closed' nature of a game is such that modification to rules is done by participants only. Thus the outsider cannot recommend what is to be taught in mathematics in schools, if mathematics is held by the teachers to be strictly a game. If one assumes that what one teaches is non-serious then one may well feel no responsibility for how elements of it required in other school subjects, are acquired. Thus one can find schools in which the science teacher teaches all the mathematics his pupils will need. One may find Nuffield Science strictly adhered to in the laboratories and S.M.P. equally strictly adhered to in mathematics classrooms. The result is that the pupil meets weight

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1. By 'privacy' is meant the notion that the language of mathematics is not coincident with ordinary language but is technical. Thus one must be introduced into the technical language to be able to participate meaningfully.

(density) in first year science, but only in the second year in mathematics.[1]

B) The objection is misdirected for what is in fact argued is that fox-hunting is not a game for it is not simply underpinned by morality, as is any human activity, but actually involves moral decisions within the game itself. Thus deciding when the fox 'has had enough' is a moral decision, and also an essential feature of fox-hunting. There would not seem to be a similar argument against mathematics for the 'serious' decisions are those made by teachers, parents and pupils <u>about</u> what is to be studied rather than <u>within</u> the studying. The only possible line is that the mathematician is morally obliged to show cognisance of the possible interpretations of his work in other fields, which may include the eventual production of weapons, say. This is again to argue that mathematics is not a game for it has interpretations, which is precisely what the Bourbakists say is the province of non-mathematicians. This is an impasse and will be set aside until there has been a discussion of the views of mathematicians who reject the Bourbakist standpoint.

<u>Objection 2</u>. To teach mathematics as a game is to deny that mathematics is a form of knowledge in Hirst's sense, for games fall into a category identified explicitly by Hirst as not a form of knowledge.

<u>Reply 2</u>. Again, two types of reply seem possible,

A) It is agreed that mathematics is not a form of knowledge in Hirst's sense but this is not to say that one cannot

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1. See the article, 'Mathematics and science in the secondary school' by A. J. Malpas, in <u>Developments in Mathematical Education</u>, pp.233-40, but particularly the diagram on p. 235.

accept alternative arguments for structuring the curriculum. One may accept, for example, the argument presented by Phenix in Realms of Meaning, that mathematics is part of 'symbolics'. All games are to be seen as 'a collection of arbitrary symbolic systems' and mathematics is one of these. Symbolics is to be part of a school curriculum and it is up to educators whether or not there are other good reasons for introducing symbolics through mathematics, rather than through chess, say.[1]

B) It would seem that one might bring this objection because one believes that games clearly occur in other areas than mathematics and only exist by being parasitic upon these other forms of knowledge. Thus a history teacher may use the game, 'Stalingrad' to discuss the reasons for the Germans losing the Second World War. In principle, the whole history syllabus could be built around such games but at no point could one claim that history was identifiable with this set of games and continue to claim, if one does, that it is a form of knowledge. Thus the argument runs, is not mathematics just like this? Mathematics is presupposed in the establishing of the games.

The reply to this is that 'games' do not occur in other forms of knowledge as they occur in mathematics. The games that occur in history or economics or whatever are 'simulation games' and the qualifying term is also a

1. In Phenix's analysis no distinctions are drawn between systems which necessarily possess coding/decoding structures and those that do not. In this way, he fails to provide any appreciation of the Bourbaki standpoint for example.

negating term. One plays a simulation game in order to learn about what it simulates, and so its raison d'être is not intrinsic. This is precisely what is not being claimed of the game of mathematics. It is simulation or interpretation which is denounced as irrelevant. Thus mathematics is a game in just the non-serious sense covered by the criteria given on p. 135 above.

Yet one may accept that Hirst is right to identify mathematics as a 'Form of knowledge', and still argue that the 'unifying label',[1] 'game', in no way contradicts this. Hirst chooses to talk of the link between 'knowledge' and 'experiences' rather than 'reality', and in so doing cannot be requiring each form of knowledge to be logically necessary for 'reality'. In other words, mathematics may be contingently connected to the serious business of life, for Hirst does not assert that a form of knowledge is to have a logical connection with 'real life'.[2]

Objection 3. To teach mathematics as a game eliminates the possibility of teaching mathematics in many other interesting ways, like 'discovery teaching methods'.

Reply 3. It is difficult to deal fully with this objection until the alternative movements, centring on other approaches, like 'discovery' are fully presented. So far the conception of 'freedom' that a pupil may be given has been shown to be relative to the objective to be achieved. That is, the pupil must have sufficient knowledge for the objective to be within his reach before he can be said to be 'free to discover it'.

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1. Hirst indicates by this phrase in 'Liberal Education and the Nature of Knowledge', Archambault, that 'game' lacks the criterion necessary for 'knowledge'. This point should be considered along with Criterion 2 on p. 135 above for 'game.

2. It is reasonable to assume that the developed mind will include awareness of abstract as well as applied problem-solving.

In this sense, 'discovery' is no less a possibility for this approach. In Learning Mathematics, Book 1, Section 4, there are many spacial games, including tanagrams and pentominoes. Once the pupil has the rules for combining five squares to form a pentomino he may be left to discover a number of the 2,339 possible combinations for producing a 10 by 6 rectangle with them.[1] The fact that the teacher knows that there are that many possibilities and that the pupil cannot call a shape joined only at the corners, a pentomino, is restrictive but does not eliminate 'discovery' in any usual sense. Similarly, the view of the Bourbakists may be that as a game, mathematics is just a matter of 'convention' but this does not necessarily remove the notion of 'discovery' unless one stipulates that it may be used only of what is 'pre-existent'.[2]

Objection 4. 'Truth' is an essential concept within mathematics but it seems to be necessarily eliminated from the view of mathematics as a game.

Reply 4. A philosophy of mathematics has been introduced already that eliminates 'truth'. That is the theory of the Strict Formalists. Furthermore, in the discussion of 'truth' in part 1 Chapter 5, the Redundancy Theory was presented which argues that no additional meaning is given by the concept, 'truth'. These strict Formalists argue that all mathematics can be adequately presented without any introduction of interpretations. 'Truth' is only necessary for 'interpreted' systems and not for the establishing of the systems in themselves. However 'true' and 'false' are employed within the Bourbaki system to indicate relationships that are demonstrated within limits. In other words, 498 + 326 = 824 is a 'true relation' in the sense that 'I' have not gone back to the first

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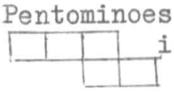
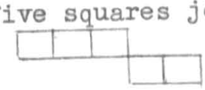
1. Pentominoes are combinations of five squares joined at the sides. Thus,  is acceptable, but not . There are 12 such combinations.

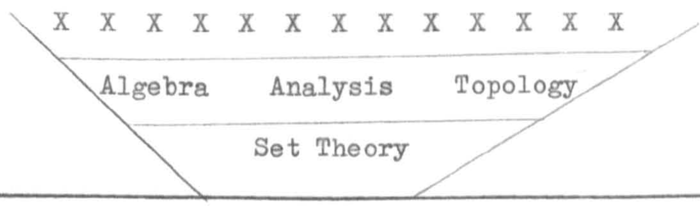
2. This is the issue that will be developed in the next chapter on the Scientific approach.

principles of Set Theory to work it out. Thus 'True' and 'False' are elements in the abbreviations of the game, but not of the game as demonstrable in principle.[1]

A final comeback may be to challenge this position over the problem of paradoxes, but this is to misunderstand the view presented here. It is not a position that stands or falls on its complete consistency; it is not a philosophical argument. In other words, if paradoxes arise then one simply modifies the original set theory to eliminate it. This is precisely an example of Lakatos' monster-barring.[2]

b) The 'family of games' view

Traditionally someone taking School Certificate in Pure Mathematics in the nineteen-forties would have expected to answer questions in three discrete areas, 'Arithmetic', 'Algebra' and 'Geometry'. Twenty years later an undergraduate in Pure Mathematics would have found himself taking courses in 'Numerical Analysis', 'Algebra' and 'Topology'. According to both standpoints, it makes sense to subdivide Mathematics. The rationale for the latter subdivision has strong support in the works of the educationist, Piaget. While Piaget has great sympathy for Bourbaki he presents his own view as a natural development of such static formalisations, consequent upon the stimulus of Goedel. Piaget is not happy to identify any system as a given foundation for anything else. Bourbaki presents a view of mathematics as a pyramid resting on its apex, 'set theory'. Thus,



1. This is not to introduce a further notion of truth for this is still the Correspondence Theory as modified by Austin, and explained in the appendix, pp. 274-5 . It may be called a form of 'truth by convention'. 'True' just acts as a sign to show that only an abbreviated proof is being given here, although a full proof is known.
2. See Proofs and Refutations, pp. 14-23, and part 1 Chapter 4 above.

Piaget says that any layer is dependent for its relative security on its links with the layer above and the layer below. All layers are themselves developing and there is no end to the layers. This briefly is the programme of Piaget's Structuralism. Within this is the idea that there are three basic forms of 'parental structures', which may be roughly identified as 'algebraic', 'order' and 'topological'.[1]

Thus mathematics may be viewed as a family of games consisting not only of algebra, numerical analysis and topology but further, of combinations of these into new games, like algebraic topology, et al. However, within the Secondary School sufficient games are identified by the parent structures themselves, together with the 'language' of their formalisation, 'set theory', to cover all that is to be taught by sixteen.

In practical terms, the result has been the possibility of identifying these structures within a modern mathematics series presented distinctly throughout the series. S.M.P. is such a series. While it wishes 'mathematics to be seen as a unified subject and not as a collection of independent topics' (Preface to Book F) the 'parental structures' are not cross-linked.[2] This view is essentially one of diffidence. It is the view of those educators keen on formalisation in mathematics but unconvinced that mathematics is unifiable in just the sense identified by groups like the Bourbakists.

c) The 'through games' view

If one identifies mathematics with the natural sciences or technology in just the way the Hypothesisors do in part 1 Chapter 4, then it would be illogical to treat mathematics as non-serious: as a game. No part of science or technology is a game. This does not prevent a teacher with

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1. These are the terms Piaget uses in Structuralism, p. 26. Piaget provides a framework for learning which is adaptable to either the 'Games' or the 'Science' approaches.

2. For example, matrix multiplication is given three applications in Book F, to networks, relations and transformations, but they all lie in one structure, 'algebra'.

such a view teaching parts of mathematics as if it were a game, or using games within mathematics.

Thus a book 'designed to build a firm foundation of understanding of the principles of arithmetic, to show how numbers are used to describe physical models, and to build a bridge between arithmetic and algebra' may end with a chapter entitled 'More mathematical mysteries' whose purpose is 'only for enjoyment. Explore them to your heart's content. You may even become sufficiently intrigued by them to make some additional discoveries of your own.' This is the contrast between the preface on p. v, and the introduction to the concluding chapter on p. 356 of an American high school text, entitled, Foundations of Mathematics (Holt, Rinehart, Winston, 1962). The overall concentration of the book is on 'physical settings' and 'physical meaning' but it is still possible to have the non-seriousness of a question entitled 'Jealous husbands'. It is essentially a reformulation of the example given on p. 103 above, using a flow-chart. However, in this example, the boat takes two adults and a wife is never to be left without her husband if another man is to be present.

It should not be assumed that 'realism' and 'games' can only go together if the teacher takes the 'through games' point of view. A game may involve the objects of reality and may even have a contribution to make to a pupil's coping with reality but it is not the objective of views a) and b) that that is why the pupils should be taught to classify shapes as squares, triangles, etc. for example. View c) is distinct precisely because it indicates the primacy of application over 'enjoyment'. Even in the example given of the jealous husbands the teacher hopes that the pupils will develop further their use of flow charts which tomorrow they could be using in Home Economics to organise their cooking practical.

d) The 'in parts' view

While one teacher identifies mathematics with 'games', another may consider some mathematical activities as games, but not all. He accepts that not all games are mathematical and vice versa. Just as there are things that one does with cards which are not 'card games': one may build a card house or teach a child to count with them. Similarly a teacher may present some mathematics as pursuable as a family of games, but the rest as applicable in other areas inside and outside school. This is reminiscent of Frege's view that arithmetic was reducible to logic but geometry was synthetic a priori. The teacher might say that there is a theory of algebra totally derived from set theory which is a closed game. Contrasted with this is the application of mathematics to 'space' which is identifiable generally as 'topological theory'.

The teacher identified above may have made a much clearer division than that found in practising teachers. This point of view may be exemplified in a kind of schizophrenia that one finds in some text books. The result is that chapters even become divided between 'games playing' and 'applicability'. That this is not indicative of view c) is made clear by the theoretical underpinning found in prefaces and introductions. For example, the series Mathematics through Experience has a title which sets one to expect a clear approach of the kind to be discussed in the next chapter. However the preface states: 'The language and notation of sets are used throughout to clarify and unify previously unrelated concepts. Emphasis is placed on pattern and structure.' Yet the Introduction informs one that mathematics 'is an activity in which we learn to organise, systematise and predict from our own experiences. The result is that games like 'magic squares', 'chinese multiplication' and 'the theory of sets' are intermingled with 'slide rules', 'timetables' and 'reading the meter' all in the first chapter of book 1.

If one has no theoretical commitments then it would seem possible to select from other views but it is also reasonable that when this is to be presented as a text book, the neutrality should be made explicit.

MATHEMATICS TEACHING, GAMES AND PHILOSOPHICAL MOVEMENTS

There is a danger that someone who is committed to the 'games approach'[1] and is also committed to the complete axiomatisation of mathematics in either the sense derived from Russell's logicism or Hilbert's formalism, will believe that one must learn mathematics in its logical order. The danger is that a pupil is required to progress rapidly from the minimal abstractions necessary to establish set theory for example, to its completely abstract use in the game of set theory. If there is a philosophical error here then it is of the highly technical kind that is found in the work of Goedel et al. This would not be the basis of useful criticism in a thesis concerned with secondary education rather than higher education. The point is that 'understanding set theory' may mean communicating formally with mathematicians in set theoretical terminology alone, but it makes no practical sense to call this, 'understanding set theory for eleven year olds', for very few such eleven year olds will ever exist. In other words, if mathematical games playing must be more abstract than chess playing say, then the mathematics teaching must involve psychological considerations. There is a point at which formal requirements and common sense conflict.

A concern for such dangers comes out most clearly in the work of Dienes. On the one hand Dienes believes, that 'most mathematical structures can be learned by playing skillfully contrived and excitingly motivating games of a mathematical nature', while on the other hand, he states that it 'makes more sense to start with the world as it appears, abstract from it, represent the abstraction, and then symbolize it,

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1. If this is taken in the sense explained in (c) above, as 'The "through games" view', this danger cannot occur.

than to start with the symbols.' See 'Some Reflections on Learning Mathematics' in Learning and the Nature of Mathematics, pp. 49-67. The point is that mathematical progress cannot be isolated from conceptual development.[1] At a particular conceptual level a child may not be capable of even abstracting from concrete embodiments of a given relationship, let alone have the understanding of the abstraction presented formally. Two relevant points arise from this particular article,

1) All 'understanding in mathematics' must be related to conceptual development.
2) Even given Dienes' commitment to 'games', he combines this with the additional educational objective that learning to draw 'abstractions' from concrete examples and make 'generalisations' from these abstractions is not just valuable to mathematical but to all thinking. This must be so because 'mathematics relates',[2] as Dienes puts it, on p. 52 of his article.

Thus Dienes rejects views a) and b) for some form of d). The argument that arises is that even if mathematics is isomorphic with a set of games, an educator who is aware that mathematics has developed because of its usefulness in other areas, cannot pretend thereafter that there are no extrinsic reasons for teaching mathematics.[3]

Now, Dienes' first point that all 'understanding in mathematics' is related to conceptual development can be best exemplified by indicating

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1. 'Conceptual development' is taken by Dienes to follow the lines presented by Piaget in for example, The Child's Conception of Number, (with A. Szeminska).

2. Dienes sees the existence of mathematics in evolutionary terms. It exists as it is, in order that relationships in the universe can be identified, using scientific knowledge as it stands at a particular time.

3. Dienes would go further and say that if one took away the extrinsic reasons then mathematics would lose its justification for 95% of the population could acquire all that they needed to know of mathematics elsewhere. See 'Learning Mathematics' in Mathematical Education, pp. 81-95.

how he sees the nature of 'understanding'. It involves

a) Generalisation[1] – a child picks up some stones and counts them. He counts 8.

the child lays them on the ground and counts them left to right and gets 8.

the child counts right to left and gets 8 and also gets 8 when they are in a circle.

the child tells his mother that 'counting is to do with what there is rather than where it is'.

This child has begun to <u>understand</u> what one means by cardinal numbers.

b) Abstraction – in a class the teacher asks one pupil to think of one or two operations that can be done to a number. Members of the class shout out numbers and are given answers until someone is able to say what the operations are. Thus 4/15; 6/35; 3/8 and a child then shouts out x/x^2-1.

This child <u>understands</u> how this series is related.

c) Proof – a class is asked to prove that 2 odd numbers make an even number. One child writes, if k and l are odd numbers then they may be rewritten as $2K + 1$ and $2L + 1$ (not divisible by 2) adding them gives $2K + 2L + 2 = 2(K + L + 1)$ which is divisible by 2 and is thus even.

This child has begun to <u>understand</u> a deductive sequence.

These do not show sufficient conditions for 'understanding' but may give some indications of its requirements. Other approaches may require a link between applicability and understanding and another may require that a pupil corrects his own errors and continues towards achieving a correct answer rather than waiting for teacher. These are perhaps elaborations on the principal conditions indicated above, although no

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1. Dienes gives lucid explanations of this term in 'Learning Mathematics', and also of 'abstraction'.

conclusive claims are being made for those given.[1]

However, if a philosophical movement is not prepared to consider as 'mathematical understanding' anything less than an appreciation of completed deductive sequences, then the phrase has little use in schools. The point would seem to be that strict formalism is only concerned with the formal manipulation of symbols according to conventionally prescribed rules and it should be clear now that what a secondary pupil may achieve in his five years, 11 to 16, may be no more than the beginnings of such abstract manipulations. The philosophical point is that the teacher committed to such a philosophy would have to admit that only a small proportion of his teaching can be with the intention of achieving 'mathematical understanding' in this sense. Most teaching would be through concrete games, and so would be barely touching the abstract.[2] The Intuitionist would be even more[3] disappointed for he requires that the pupils produce completed proofs before he can know that they have achieved any of the mental constructs that he calls mathematics. There is clearly a sense in which Intuitionism's concern for mental constructs conflicts with the fundamental nature of this approach which is with conventionally produced games. However there is some common sympathy, for Intuitionism claims an autonomy for mathematics which is echoed in some of the views discussed earlier in this chapter.

Of the other philosophical positions it has been argued already that there is a natural compatibility of beliefs between this games

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1. The term 'mathematical understanding' is further discussed in part 3 of the thesis.

2. Dienes certainly supports the formalised nature of mathematics as central, but argues that children can pick out 'general rules' for improving their chances in a game, before they are conceptually ready to accept the formalised language in which it would be put in mathematics. An interesting introduction to these points occurs in <u>Let's Play Maths</u>, pp. 13-28.

3. One might express this as: the Strict Formalists are demanding in the area of mathematical rigour, but the Intuitionists are even more demanding.

approach and Logicism and Hilbertian Formalism, but a natural hostility for this approach from the Hypothesisors. However a position not tempered by Dienes' intervention would have the same difficulties as those just shown for the philosophical position of the Strict Formalists. Equally Dienes' own position has a feature strongly reminiscent of hypothesising when he identifies 'generalisation' with 'what will happen if?'. That is, a generalisation occurs when one generalises the results that one has at present. Dienes is willing to talk of a pupil 'doing mathematics', before there is any guarantee that he has grasped the idea of 'abstraction'.

SYLLABUS ORDER

Naturally if one were still to try to hold to views a) and b) then there would be critical determinants[1] of the syllabus based upon acceptance of the logical development of mathematics as a whole (view a) or of algebra, analysis and topology (view b). However, logical order is not the only criterion to be considered. Another is the facilitating of the development of those ways of thinking so characteristic of mathematics. This is to move the focus of mathematics teaching further away from 'mathematics' itself, and concentrate its effects on pupils. This is not an open cheque, for mathematics is still to be seen as a formal system and as such, 'understanding' is only a possibility if the step towards which the teacher aims at any time are all attained. That is to say that if one decides that 'Pythagoras' Theorem' is worth teaching then necessarily the pupils must learn at some time previously what an angle is, and so on. In this sense, the subject must necessarily retain a grip on the order of what is taught.

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1. For Logicists, Formalists and Formalisers like the Bourbaki, there is no logically possible entry to mathematics (or some sub-system) except through the foundations that are logical axioms, self-evident axioms or set theoretical axioms, according to one's standpoint. Understanding the foundations is a pre-requisite of having the right to claim that one has 'mathematical understanding'.

CONCLUSION

The games perspective indicates at least one criterion that sets it apart[1] from any other to be discussed. Through the concept of a game there is the concern to do mathematics just for its own sake. While Dienes' limitations on this for the teacher's objectives may have to be taken into account, the pupils will be expected still to find mathematics 'fun'. The arts too may be fun, but the concentration there is upon achievement of an end product, but this is not what is emphasised here. The emphasis is upon the manipulating rather than on the establishment of something at the end of the manipulation. 'Ends' are logically necessary but they come as a matter of course if the manipulations are correct. To focus on 'manipulation' as the basis of mathematical attainment, is to select as objectives, features of mathematics that are logically independent of success, achieved in other subjects. This provides a natural source of 'isolationism'. The pleasure that seems so important to this perspective must involve some kind of aesthetic response to the mathematical situation. This presupposes knowledge of some mathematics and some on-going use of the language of criticism.[2] It does not necessarily imply that this language was learned in the viewing of mathematics or any part of it, as works of art. Of course under view d) this would be a possibility, but it ought not to be taken as necessary.

Finally, in considering the problem of syllabus order there has been the reiteration of the point made for the art perspective that

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1. This can be seen as the reply to question 1, given on p. 107 above, 'What distinguishes the games perspective from all the others?'

2. For example, one learns $5 + 3 = 3 + 5$, $6 + 7 = 7 + 6$,...what happens if given $27 + 98$, one recognises that $27 + 98 = 98 + 27$. That is, one acts according to the general rule, $a + b = b + a$. The problem of 'knowledge without language' has been touched upon in the discussion of prerequisites of aesthetic criticism on pp. 121-23 above. The issue is too large for handling here.

understanding some proposition in mathematics logically requires a knowledge of some previous propositions tied logically to the present objective. In that sense, if one can come to agree on endpoints across the perspectives, then there will necessarily be some overlap in the syllabuses as a whole.[1]

1. This issue is central to Chapter 11 below.

CHAPTER 9

MATHEMATICS TEACHING - SCIENCE PERSPECTIVE

INTRODUCTION

No less has been written about science by philosophers in the last hundred years than about mathematics. For every mathematics movement that one might identify there would be at least one scientific movement. It is not the intention of this chapter to carry out a brief survey of all these views of science. The term 'science' can be used broadly to refer to any 'seeking after knowledge', whether it might be the 'science of crossword-solving' or 'science of cock fighting' at one end of the scale, or precisely to refer to 'seeking knowledge of the nature and behaviour of the physical and social world by the construction of conceptual frameworks whose propositions are amenable to empirical refutation', at the other end of the scale. An attempt to cover such a scale of usage would be both impractical and unnecessary. It is sufficient here to identify that there is this breadth of common usage, and that it may be necessary in the discussions that follow in this and the next chapter to refer to the dangers of ambiguity that can arise from the breadth of usage. Two epistemological standpoints will be employed extensively in this thesis, and so these will be looked at in some detail in this chapter. For want of better names, one may call one, 'the Newtonian view' and the other, 'the Evolutionary view'. That they are epistemological views is explained by the fact that the Newtonian view is characterised by the belief that, 'man can know that he has discovered how the world really is', and the Evolutionary view, characterised by the belief that, 'man can know that a given description of the world is better than a previous one'.[1]

A mathematics teacher who is concerned with children acquiring a model method will be influenced by either of these views.[2] There will inevitably be some suggestion that a teacher who believes in the eternal truths of mathematics will feel more at home in the Newtonian view, while someone with a synthetic a posteriori characterisation of mathematics will choose the Evolutionary view. Of course, as has been mentioned before, philosophical considerations are not the only ones that determine how one teaches, and so philosophy of mathematics cannot be tied logically to how one will actually teach.

It is being suggested that the perspective considered here is that which concentrates on the method of science and this should not be confused with the perspective that will be considered in the next chapter. In that chapter, the focus is upon teaching mathematics through its use. That is, it is science, if science is 'helping man to deal with his environment'.

This chapter will attempt to clarify the above two scientific views through their exemplification in mathematics teaching and consider the three questions identified at the end of part 2 Chapter 6.[3]

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1. The word 'better' is certainly contentious but hopefully is clarified by the end of the chapter.

2. It is not claimed that there are not other relevant views, notably a Revolutionary view found in the works of Kuhn and of Feyerabend who argue that theoretical changes are 'radical' rather than 'evolving'. There is also the Marxist-Leninist view attacked in Popper's works. Support for it is found in N. I. Styazhkin's History of Mathematical Logic from Leibniz to Peano. No doubt, both views have the individual support of mathematics teachers, but not in the form of a movement, yet.

3. The questions were: 1) What distinguishes perspective X from all the others? 2) Must one logically have come to appreciate one perspective before, as a pupil, one can appreciate some other perspective? 3) Does one find a common logical sequence to what is taught, no matter through which approach it is taught?

TWO VIEWS OF SCIENCE

From late in the sixteenth century to early in the twentieth century, science was given a status that had previously only been given to religion. However the monuments built by Galileo, Harvey, Newton and others were shattered by the work of Darwin, Maxwell, Einstein, and Born. Today monuments are built one second and shattered the next. The result has been that the confident presentation of theories is replaced now by the diffident presentation of models.[1] Thus, at least two methods of science have had time to infiltrate mathematics teaching.

1) The Newtonian View

Newton wrote, 'The main business of natural philosophy is to argue from phenomena.' (*Principia*, preface). This indicates one feature of what came to be believed as *the* method of science, *Induction*. This is the making of a generalisation from a number of particular observations. However the inductive generalisation is only a scientific Law, once it indicates why the generalisation has been found. In other words, one might find that whatever an object's weight, it hits the ground at the same time. This is not a law for it does not explain *why* one saw all these objects hitting the ground at the same time. It only becomes a scientific explanation once one learns that 'weight' itself is to be explained in terms of a special force, 'gravity'. Given the explanation, other problems may be resolved by deduction alone. This is called the *Hypothetico-deductive* method. Newton used this method in the *Principia* to explain the motions of all the bodies then observable in the universe. Finally, under what has been called here the Newtonian view, a theory that explains all related phenomena is taken to be true in an absolute

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1. See the distinction between 'theory' and 'model' given by J. C. Forge in 'A Role for Philosophy of Science in the Teaching of Science' in *Journal of Philosophy of Education*, Vol. 13, 1979. A model is taken as a useful but not a true description, while a theory is believed to be a true description.

sense. It is thought to provide propositions that would <u>correspond</u> eternally with reality.[1] As the view is found in both Leibniz and Kant,[2] the laws of nature could be other than they are. Yet they are not man-made but pre-existent. For example, Newton's fundamental laws of mechanics are <u>a priori</u>, in just the sense identified on p. 76 above, 'disconnected from <u>changes</u> in the world', and therefore not open to falsification by the phenomena, once the law is identified.

2) <u>The Evolutionary View</u>

It has already been noted in part 1 Chapter 4 that Peirce was committed to the 'Method of Science'. However he realised that the status of scientific laws as identified by man could not be known to be <u>the</u> laws of nature, even if one believed, as Peirce himself did, that there was that kind of order in reality. Thus a principle of Peirce's method is 'fallibilism'. Any given theoretical position is liable to later modification, but Peirce believed that a series of modifications leads inevitably closer to the 'truth'. One more important alteration that Peirce makes to the Newtonian method is that he sees 'induction' as a second stage in science. The first stage, he calls 'abduction'. This is the point where one has a few observations that one feels are connected under some relation but which one is not prepared at that time to assert as an hypothesis. In more contemporary language one might say that one makes 'conjectures'. Psychologically their status is important, in that one does not feel that their refutation or modification is a big blow to one's pride. Eventually the conjecture hardens and one tests it against

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1. See the definition of the Correspondence theory of truth given.

2. The seal of God on man's knowledge of <u>the</u> laws of nature is argued for, by both. See P. P. Wiener's <u>Leibniz Selections</u>, pp. 65-70, 152-6, and 539-47; and in Kant, <u>Critique of Pure Reason</u> (A125-127 for example). Kant sees mechanics as an example of 'pure natural science' which includes geometry and all areas based upon synthetic <u>a priori</u> principles, but excludes such sciences as chemistry and biology which Kant did not recognise as having such fundamental principles underlying them.

a number of particular instances. If the conjecture holds then one has a 'hypothesis' which has been achieved through 'induction'. Finally one resolves other problems by drawing deductive connections from the new hypothesis, and then testing these out against reality. This view of science is clearly much more elaborate and less secure than the Newtonian. It may be roughly identified as a five stage method:

1) Conjecture
2) Observation under the conjecture
3) Induction
4) Deduction
5) Constant willingness to modify (fallibilism).

VIEWS OF SCIENCE AND MATHEMATICS TEACHING

1) Taking the Newtonian View a teacher of mathematics may well feel that this 'old-fashioned' view of science is precisely the approach to mathematics that he wishes to encourage in his pupils. Furthermore, it seems to take cognisance of the problem noted on p.150 above that children in secondary schools are unable in the main, to take in new abstract ideas. Dienes' 'abstraction' seems to have much in common with inductive processes in science.[1] Similarly his notion of 'generalisation' is well exemplified by the way Newton took Kepler's work on planetary revolutions, to produce a generalised theory for all bodies.[2] In this way one may identify a pairing of 'induction' and 'abstraction' and 'hypothetico-deduction' with 'generalisation'.

A further attraction is the fact that a teacher can identify this scientific model as claiming *a priori* truths, which may be precisely what the teacher may want to see in mathematics. It provides a model reminiscent of Russell's, in which 'abstraction' is followed by 'deduction',

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1. This may be even nearer to Peirce's notion of 'abduction', but that does not exclude the possibility of this linkage too.
2. See Hanson's discussion of this in Patterns of Discovery, pp. 72-85.

and which thereby confirmed Russell's essential empiricism.[1] One has a method that explains the progress from concrete beginnings to formal and eternal conclusions.

Signs of this approach are to be found in secondary school text books written just after the war. Series like Mathematics for Modern Schools took on a more concrete approach rather than a 'formal' approach, but stopped short of allowing children to make their own abstractions. Thus in Book 1 of this series, p. 166, one finds the following example: the book is introducing corresponding angles and has already introduced parallels through railway lines. However it is indicated that it would be more sensible to focus on objects that one can have in the classroom, 'if anyone has a train-set one section of straight track will do...' Then one moves to the next level of abstraction and measures the angles of drawn lines. 'Using a protractor, measure the angles x and y which are in the "corresponding angle" position. Shift the straight edge several times, and each time measure the angles in the corresponding angle position. You should find that the separate pairs are always equal to each other.

By doing this you have checked the truth of this very important statement...' The pupil is almost encouraged to follow the method of science and two pages later on p. 168 he will prove that alternate angles are equal - the hypothetico-deduction of the Newtonian Method.

Twenty-five years later one finds in Mathematics in Primary Schools a total commitment to children making 'real discoveries' and the view that 'science and mathematics should be very closely associated' for they both involve 'a discovery of relationships' (see pp. 141-42). This is to identify mathematics with abstract relationships to which the concrete experience is only a preliminary, for facilitating understanding.

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1. See the discussion of Russell's views on pp. 33 to 36 above.

What is interesting, given some recent comments on 'discovery in Nuffield science',[1] is that the older text is keen on concrete experiments, not on pupils ever thinking that what they are doing is anything more than 'rediscovery', rather than 'real discovery'.

In actual practice a teacher may not hold one hundred percent to his pupils studying mathematics in this way. With the previous chapter in mind[2] it would seem to be the case that four positions are identifiable. These are,

a) Mathematics is a science and pupils are to be taught it as such.

b) Mathematics is divisible logically, for example as Frege believed, and one part is to be taught as a science and the other part as a game, say.

c) For pedagogic reasons, at least some parts of mathematics are to be taught through the method of science.

d) For pedagogic reasons, all of mathematics is to be taught through the method of science.

2) <u>Taking the Evolutionary view</u> a teacher of mathematics may well feel that this provides a picture of the actual achievement of most secondary school children in mathematics. They do not ever reach a point when they would believe that one could hold what they are said to have learned as eternally true. This is not to indicate that they have doubts about '2 + 2 = 4', but they may have doubts about '$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$' as the general solution of the quadratic equation, '$ax^2 + bx + c = 0$'.

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1. Stevens argues that there are dangers in presenting 'teaching about science' as 'teaching to do science', when the discoveries are heavily guided into almost always being 'rediscoveries', which is not science at all. Stevens says that this may lead to disenchantment, once pupils realise that 'real discovery' is not wanted. See 'The Nuffield Philosophy of Science' in <u>Journal of Philosophy of Education</u>, Vol. 12, 1978. (See particularly p. 103). An incidental point is that both these 'progressive' projects and <u>Mathematics for Modern Schools</u> use 'we' to reinforce a partnership in exploration?

2. See the four views given on p. 137 above.

In this somewhat negative justification, one tells the pupils that they should not worry about having doubts because great mathematicians have them too. This may be right but a further justification, or perhaps a rewording of this justification may be, that mathematics is not a dead corpse that is gradually uncovered but a living animal that grows bigger every day. This, in a sense, is to look at the evolutionary approach from a different aspect.

This is the view that comes through in the work of Raymond Wilder. It is well congested in his article, 'The Nature of Modern Mathematics' in Learning and the Nature of Mathematics, pp. 35-48. Wilder argues that mathematics is a science and not just a formal game that began through abstractions. He sees abstractions as on-going in the continual feedback of information from the continual attempt to provide 'better' explanations of reality.

> ...as mathematics evolved, a higher order of abstraction was achieved, in which concepts began to be applied to concepts. Mathematics gradually added concepts of its own to the world of reality, so that its domain of application included not only the physical environment, but the cultural...no matter how abstract and seemingly removed from physical reality mathematics may become, it works; it can be applied...Even though the dual nature of mathematics may seem to split it into one part that can be applied and one part that seems to be just something for professional mathematicians to play with, there is actually no separation. Both aspects of mathematics serve a scientific function,...(every mathematician) is a participant in the evolution of mathematics, whether he likes it or not. If he tries to exile himself and play mathematical games having no relation to the world of reality, he will not be heard. (Pp. 41-2).

Wilder emphasises the idea that mathematics begins and ends in the real world. Between these points there is always the possibility of new methods and new concepts. That is, today one proves Pythagoras by one construction and tomorrow by another, and on the next day by Trigonometry, rather than Euclidean geometry. In this sense, there is a far more realistic possibility for 'real discovery' in mathematics than in the natural sciences and a teacher who insists that there is only one way, is not just teaching badly but is ignorant of the nature of mathematics. This is the essence of Wilder's position. Philosophically, it rejects logicism for being 'dead', and intuitionism for being overprotective. It is rather similar to the Wittgensteinian position outlined in part 1 Chapter 3 above that seeks a route between formal axiomatisation and constructivism. The Wilder position has reflections in actual text books. In the preface to the appropriately named Discovery Mathematics series, one reads 'Mathematics is not a static thing, it is progressive and plays a vital part in our lives and we should think of it in these terms when we are teaching.' One finds that in this series pupils are left with open questions, rather than with the questions being left open for one or two lines only, as in the earlier Mathematics for Modern Schools. 'Abstraction' and 'generalisation' have strong presence in the Discovery Mathematics series. Compare 'what makes a square square?' on p. 7 with 'what would happen if your books were odd shapes?' on p. 9 of Book 1.[1]

It is not easy to find examples of practising mathematics teachers who follow Wilder, for in recent years they have taken this evolutionary view on a stage, to a concentration on the applicative value of mathematics which is to be considered in the next chapter. However with this in mind, it may be worth considering the view of mathematics given by one of those leaders of practical progressivism in mathematics education, W. W. Sawyer. Sawyer identifies five stages in the discovery of a mathematical theorem.

'(1) Someone has to suspect the truth of the theorem. (2) Someone has to discover why it is true, i.e. to discern the kind of argument by which it may be proved. (3) The argument has to be embodied in a

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1. It would be wrong to assume that the view presented here is that the series holds exclusively to this 'Dienes-like' approach, for in fact it also includes some traditional pedagogic techniques such as identifying 'Make "top heavy"...Turn divisor upside down and Multiply...' (Book 1, p. 89).

formal proof. (4) The theorem has to be remembered. (5) Fruitful applications of it have to be found.' ('The Role of Intuition in Mathematics Teaching' in Graduate Training of Mathematics Teachers, Canadian Mathematical Congress, 1972, pp. 63-75). Certainly this is not too unlike the sequence of 'Conjecture, Observation, Induction, Deduction and Fallibilism' identified for the Evolutionary View of Science suggested on p. 161 above. One might argue that the Fallibilism stage must logically presuppose that the hypothesis is remembered, so that disconnection may be accounted for. Similarly the move from 'conjecture' to what Lakatos calls 'informal proof'[1] is covered by Sawyer's rewording of (2) a few lines later, as 'casting around for means of proof'. This suggests at least that the movement from (1) to (2) is far from immediate, if it involves 'casting around' ('observation'?). Another such analysis is provided by someone else who has influenced all levels of mathematics education, H. B. Griffiths. He presents a four stage progress to a mathematical theorem. His stages may be summarised as, (1) a guess; (2) 'insight' into 'some "right" kind of way'; (3) checks; (4) formulated proof (See 'The Structure of Pure Mathematics' in Mathematical Education, p. 18). This is surely again very reminiscent of Sawyer's first three stages, the first four stages given on p. 161 above and the whole ethos of the work of Peirce. Griffith makes the reassuring comment that although 'the Bourbaki members will all have been through those stages...only the last one shows. Thus we see what a false notion of

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1. Lakatos uses the terms 'informal mathematics' for what precedes axiomatisation, and also for what is generally called 'metamathematics'. He also uses the terms 'informal theorem' and 'growing theorem' interchangeably. Thus, 'The theorem does not always differ from the naive conjecture. We do not necessarily improve by proving. Proofs improve when the proof-idea discovers unexpected aspects of the naive conjecture which then appear in the theorem. But in mature theories this might not be the case. It is certainly the case in young, growing theories. This intertwining of discovery and justification, of improving and proving is primarily characteristic of the latter.' (Proofs and Refutations, p. 42). It is to be hoped that this sounds largely as an echo of the main text.

mathematics is given to pupils in schools and universities, if syllabus-designers in their enthusiasm for Bourbaki's treatise are concerned only with the fourth stage of the mathematical activity...' (ibid., p. 19).

The argument presented here is not that one must logically hold that the activities identified by the above method are pieces of mathematics but that by presenting pupils with nothing but what Griffith calls, 'a Tidy System', pupils falsely consider that theirs is the only 'Untidy Activity' that occurs in mathematics. If one does come to hold the philosophy of mathematics presented in part 1 Chapter 4, then it would seem odd to present only the driest parts of mathematics to pupils, but not self-contradictory. The argument only acquires such logical force if one accepts that someone teaching mathematics desires his pupils to gain 'knowledge and understanding'. One may then argue that 'mathematical understanding' entails 'abstraction' at least, and is just one of the stages wiped away by the total emphasis on one particular stage of 'deduction'.

PUPILS AS MINI-MATHEMATICIANS

If this thesis were related to 'science' rather than to mathematics, one might reword things in the form of a question, 'can pupils make discoveries?'. The answer to this would probably be that it is logically possible, but remembering one of Sawyer's stages, one may doubt whether anybody would 'remember' that a discovery had been made. The point is that if no one is looking out for 'real discovery' then anything unusual will be treated as 'a monster' to be barred.[1] All that remains for the pupil is 'guided rediscovery' which is in no way existent in science itself. In science there is 'discovery' and there is 'replication' but

1. This is the term Lakatos uses to indicate how 'old' theories are preserved. Possible counter-examples are treated as unacceptable to the given system, 'monsters'. Stevens provides a practical example of monster-barring by a teacher over a pupil (loc. cit., pp. 105-8).

no one seems to think that 'replication' would motivate science pupils.

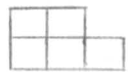
This seems particularly odd once one turns from science to mathematics. Surely 'replication' is like breathing in mathematics. Someone asks what is 17 squared, and one responds 289. One has a doubt and so 'works it out again a different way'. This is 'replication when one doubts oneself', but there are times when one 'checks' what someone else has said or even what is in the book. Calculus is full of such situations. Pupils are asked to check from first principles that $\frac{dy}{dx}$ of x^3 really is $3x^2$. Someone tells a friend that the differential of tan x is $\sec^2 x$ and not being able to imagine from where the square comes, he works it out for himself. Similarly one reads in a book, 'Given that $\sin^2 A/2 = \frac{1}{2}(1 - \cos A)$ then...' and one's reaction is that it cannot be right and one checks. 'Replication' is never made to hurt in mathematics. One may even use it as an introduction to statistics, by getting one child after another to do the same addition sum quickly, and see how the answers form a Gaussian distribution (the normal curve). The point is that if 'replication' is typical of scientific methodology, then it is certainly an activity, which pupils in mathematics can also undertake.

The harder question remains, 'Can a pupil make mathematical discoveries?'. One way out here as with science is for the project organiser to say that by discovery he means that the pupil finds out something for himself. It is a discovery for that child. This has been dealt with already, in the footnote to p. 163 above. Up until now the discussion has been about theorems in mathematics and it has been indicated that new theorems may involve long gestation. Therefore, what could be produced in a double-period is likely to be no more than a conjecture. However, if one allows the object of discovery to be broader, or perhaps more accurately narrower than a theorem, then possibilities may genuinely arise. The puzzle on p. 145 above had 2339 solutions, and while this has probably had all of them produced many times, there may be other such which have not.

At this point it may be useful to consider again, whether or not restrictions will be put on the use of the word 'discovery', by the philosophical movements discussed in part 1. For the Logicist or Formalist, it would be sufficient to have discovered how one would identify a false candidate for the pentomino problem. One would not need to write down all 2339 solutions. This in itself would not be mathematics. For the Intuitionist to be satisfied that a discovery had been achieved, one would have to show how in principle all 2339 solutions could be produced. Producing the solutions would be incidental to the real mathematics: the construction of a method for producing solutions (a recursive function). Finally the Hypothesisors would find the whole problem artificial. Yet beyond this they would not require more detailed solutions than the other movements. However if any one or more of the solutions could be shown to have an applicative value inside or outside mathematics, then this would be both a genuine and a mathematical discovery. For example, one might find that the solution outlined a possible circuit diagram.

According to the differing movements, mathematics is identified with logic or formal systems, or mental constructs, or possibilities, or even relationships, but not with the production of particular solutions. Identifying a particular 'x' as belonging to some class, or as exemplifying one member of some class is a stage of 'abstraction', and may be considered a part of mathematics at most, but cannot be sufficient to gain the title 'mathematical discovery'.

The value of this discussion has been that it has identified what is required for someone to be able to say that they have made a discovery. Two forms of discovery are possible. There is the production of a well-formed theorem on the one side, and on the other, short of this, is the production of a new method of verification, falsification or production according to the philosophy of mathematics.

These points may be identified in a simple example, to return to pentominoes again. Consider the problem of producing squares from one pentomino of the following shape  . Three people produce three different kinds of solutions to the problem.

Solution A. A builds up shapes with the pentomino and eventually finds he has made among other shapes a 10 unit by 10 unit square. When questioned, A has no idea whether or not there are any smaller squares that he has missed. This is a paradigm 'particular solution'.

Solution B. B builds up shapes and comes to the conclusion that the only rectangle that can be produced is 5 units by 2 units or a multiple of such. From this, he is able to produce a table of rectangles: 10 unit rectangle, 20 unit rectangle, 30 unit rectangle, ... 100 unit rectangle.... From this table he identifies the smallest square number as 100, a 10 by 10 square is thus constructible. B concludes that he has provided a method for producing the sequence of squares constructible from this particular pentomino.

Solution C. C takes B's work through to a formal proof.

The solution given by B may be realistic in terms of pupil possibilities but it falls short of the formal requirements of 'professional' mathematicians. It has the value of a generalisation but not that of an explanation, to reiterate the point made of the Newtonian view of science on p. 159 above. The method produces all the correct answers but B does not possess a formal explanation of its success.[1]

Pupils may be expected, particularly at 'A' level, to understand the formal explanation provided by C, but this is significantly less than expecting such pupils and perhaps even younger ones, to produce alternative proofs. One does not expect an average pupil to be a 'mini-mathematician',

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1. See a similar three-fold distinction into 'Instrumental, Relational and Logical Understanding' by R. R. Skemp, 'Goals of Learning and Qualities of Understanding' in Mathematics Teaching, 88, September 1979, pp. 44-9.

but the teacher may expect the pupil to achieve

a) an appropriate level of appreciation of mathematics,

b) the ability to replicate[1] proofs, and

c) the ability to make naive conjectures.

Unless it is made clear that c) is an essential part of mathematical progress, pupils may come to believe that in understanding formal proofs, one has grasped the whole of mathematics, and how it develops. Where teachers identify the method of mathematics with either the Newtonian or Evolutionary methods of science, as defined above, the pupils ought to become aware of the stages of development prior to rigorous formalisation. This leaves open as a question of philosophy of mathematics, whether or not the informal stages are themselves parts of mathematics. Clearly, for the Logicist and Formalist they are not. While the Intuitionist would make a similar separation, the separation as a matter of fact may not occur in public. To the Intuitionist, a proof is analysed publicly through Intuitionistic logic, but is constructed in the mind where only the person himself knows where conjecture ends and formal construction begins. The Intuitionist teacher may well make a clear-cut division between objectives a) and b) above, by identifying a) with 'appreciation of mental constructs'[2] and b) with public 'proof analysis'. For the Hypothesisors, the Newtonian method is not a satisfactory theory of mathematical method for it leads to claims of infallibility, in contradiction to the philosophy of the Hypothetical. This does not prevent its use as a model for pedagogic reasons. For the Hypothesisors, the Evolutionary method is an adequate theory of mathematical method, provided it is remembered that in this philosophy, 'method' and 'purpose' are inseparable.

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1. By 'replicate' one includes the ability to check the proof and not just repeat it.

2. On pp.117-27 above, it was argued that the route to appreciation must be through the public proof analysis, for these are generally the only 'clues' existent, to the nature of the mental construct.

To <u>conclude</u> this section, it will be necessary first to distinguish clearly, 'mathematics' from 'mathematical development'. For <u>Logicists</u>, <u>Formalists</u> and <u>Intuitionists</u> the Newtonian method is acceptable as identifying the method of mathematical development, but is viewed as a method that extends beyond mathematics itself. That is, mathematical development is all that is involved in a new mathematical theory, while mathematics only begins with the formal proof of the theory. On the other hand, the Evolutionary method of science is an acceptable identification of the method of mathematics, as described by the Hypothesisors. Under this philosophy of mathematics alone, is the method of mathematical development inseparable from mathematics itself.[1]

In following this science-approach to mathematics teaching one's main objective is not the production of mini-mathematicians, but enabling a pupil to retrace the stages of mathematical development in so far as the pupil's particular level of conceptual development allows. Most pupils only achieve an understanding of the formal nature of mathematical proofs at the end of their secondary education, and so the personal production of such proofs must be a stage beyond this.

<u>THE SCIENTIFIC METHOD AND MATHEMATICS AS A SCIENCE</u>

The intention here is to identify for each philosophical movement, the logical limitations of mathematics teaching with regard to a science perspective. In this way an organised picture of mathematics teaching in relation to the science perspective can emerge, before considering the questions posed at the end of part 2 Chapter 6 and reiterated on p. 158 above.

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1. Mathematics itself begins with 'observed facts' and ends in 'practical effects'. This is the position presented throughout part 1 Chapter 4. However, mathematics does not include propositions that depend, for their verification, on the nature of individual, empirical objects, as the more precise type of definition of 'science' noted on p. 157 above, would.

1) Logicism as described by Russell may be seen as beginning in empirical abstractions and ending in eternal truths. This matches well with the Newtonian view of science, but is in clear conflict with the Evolutionary fallibilistic viewpoint. It implies that a teacher cannot be both a logicist and believe at the same time there are no *a priori* truths, apart from identity statements. Once one accepts that mathematics consists solely of *a priori* propositions then the critical point to be made clear to pupils is that they may identify mathematical propositions by abstraction, but can only prove them by deduction. Such abstraction is consistent with logicism, for mathematics is true of all worlds including this. However mathematics is not an empirical science and cannot be taught as such. It might be taught as the science that underlies all reasoning,[1] but not as explanatory science, telling one why certain relationships occur in the world.

2) Formalism is according to Curry, presenting the thesis that mathematics is 'an objective science...dealing with a certain subject matter; ...(whose) propositions are true insofar as they correspond with the facts' ('Remarks on the Definition and Nature of Mathematics' in *Philosophy of Mathematics*, p. 152). However the 'formal systems' that make up that subject matter are totally different from the subject matter of all other sciences, for those of mathematics are man-made from beginning to end. It is a science only in the sense that it consists of a system of synthetic *a posteriori* propositions; empirical ones. Given the discussion in the previous chapter, the argument would be that a science of internal conventions is more appropriately called a game. Yet even if mathematics is not a science in the terms of either the Newtonian view or the Evolutionary one, it may be still taught through such a method. That is, within the game one may employ such a method

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1. In the broad sense, noted on p. 157 above.

to judge which move to make. Although pupils may feel themselves within the safety of a non-serious activity, they may be still encouraged to make conjectures, and so on. Furthermore, a pupil has to keep on learning new rules, and as a matter of fact these might first be learned outside the game. For example, small children learn tables as songs or learn about equality through weighing cake ingredients. To a Formalist teacher, this process would consist of informal learning by the scientific method outside mathematics, followed at an appropriate time by formal acquisition within mathematics.

3) Intuitionism identifies mathematics as an autonomous activity of a person's mind with objects dependent upon the human imagination. Hence, if science refers to reality then mathematics is not a science. While mathematics need not employ a scientific method, it is not contradictory for it to be constructed, at least on occasions, according to such a method as that presented in the Newtonian View. Further, one might suggest that Intuitionistic logic could be consistent with the Evolutionary method, for the logic is fallible, even though the mental constructs are not. Admittedly, any 'observations' made in determining the accuracy of such a logic would be back to the mind, rather than out to the world. As 'concrete intuitionism' is self-contradictory, for what is 'concrete' is not the possession of one mind, then it is difficult to see the place of Intuitionist constructs in the secondary school. The pupil acquires knowledge of an empirical study, the public form of mathematics. There are no restrictions on how this is to be learned, provided it is not presented as a system of propositions, all of which are either provable or not. The Intuitionist teacher must wait for the conceptual development of his pupils for them to do mathematics, rather than study interpretations of past constructions.

4) Hypothesisors claim, like the Empirical Formalists, that mathematics is a science. To Peirce and to Ormell, it is the 'science of possibilities' and to Lakatos, the science of 'Proofs and Refutations'. Following both Formalists and Intuitionists, mathematics is seen as a human activity but it generates much more than man initiates.[1] The mathematician is for ever finding new relations in a subject matter that he thought he had fully analysed. These are the 'discoveries' of mathematical science, but their subject matter is not that of an empirical, but that of a fictional world, Popper's 'World 3'.[1] The method employed in this activity is well-defined by the Evolutionary View, but it does not imply an identifiable, infallible method for that would be a contradiction in itself, if there are no absolutes besides trivial logical truths. Thus one may teach mathematics as a science, according to this philosophical position, but as its followers are found in practising mathematics education, they generally demand more of mathematics than being 'a pure science'. As was noted of Sawyer on p. 166 above, 'Fruitful applications of it have to be found'.[2]

THREE QUESTIONS

1) What Distinguishes the Science Perspective from all the others?

This chapter has been distinctive because of its concentration upon mathematical progress and discovery. This is indicative of the method of science as concerned with the search for 'new' true propositions, in this case, of mathematics. The concern with 'progress and discovery' contrasts with the focus in the Arts perspective on 'creating end-products/formal proofs' and in the Games perspective, on 'manipulative skills/deductive

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1. See p. 65 above for further clarification on these points.
2. Sawyer is just one of the mathematics educators who have shown great concern for applicability. Griffiths and Wilder have been mentioned in this chapter and one finds this also, not only in Ormell's text books but also in P. F. Burns, Daily Life Mathematics (1952) and P. Kaner, Modern World Mathematics (1969) for just two further examples. This is the major concern of the next chapter.

procedures'. To reiterate Sawyer's criteria, given on pp. 165-6 above, this perspective highlights how, 1) Someone comes to suspect the truth of the theorem, while the Games perspective highlights (2) the deductive procedures required to establish the proof, and the Arts perspective highlights (3) the embodiment in a formal proof. It is to be expected that the Technological-orientation will highlight (5) fruitful applications of the theorem. Sawyer notes that (5) only occurs if (4) the theorem is remembered.

2) Must one logically have come to appreciate one perspective, as a pupil, before one can appreciate some other perspective?

A pupil entering secondary school necessarily has a mathematical history. He does not have a tabula rasa with regard to mathematics. The secondary school teacher can know what the pupil has been taught in the Primary school but it is not practical for him to know by what method every pupil has been taught. This question asks one to consider whether or not it really matters that one does not know the primary school perspective(s).

It should be clear from this chapter and the two previous ones that a teacher may suggest a perspective for mathematics or a part of it, that does conflict with that presented earlier by another teacher, even in the same area of work. One teacher may introduce 'angles' axiomatically as 'rotations' (See SMP Book A, Ch. 3) within a games perspective, while another introduces them as 'facts' to be discovered through a science perspective (See Discovering Mathematics, Book 1, pp. 12ff.). Yet another stresses the fruitful applications of 'angles' for carpenters, wheelwrights and designers (See Daily Life Mathematics, Book One, pp. 154-64). Such diversity of approaches presented to a single child may lead to confusions, and certainly teachers ought to be aware that this can arise. However, nothing has been found so far to suggest that one perspective has a logical priority over another. As a matter of fact a particular philosophical form of a given perspective may be excluded at one time, for the pupils lack a sufficiently advanced level of conceptual development. In that situation the teacher will have alternatives. The pupils are to learn without understanding; or, the teacher modifies his philosophical stance to retain the perspective; or, the teacher presents to the pupils what he considers preliminary knowledge of the topic, that will facilitate acquisition of mathematical knowledge when it becomes contingently possible. A teacher realises that his pupils are too young to appreciate the insight required to produce Pythagoras theorem and so gets them discovering the relationship through cutting up squares. He does not consider that this 'really' is mathematics but believes it will facilitate later learning.

This kind of situation will be considered most critically when the aims of mathematics education and their implementation are focused on in part 3 of the thesis. It is indicative of the problems that arise, when a mathematics department sets about planning its course to C.S.E., for example. A model of the results of such a discussion was presented in the image of the young teacher, in part 2 Chapter 6. He goes from a Games perspective in Year One, to a Science Perspective in Year Two and so on (See pp. 101-07 above). The tendency in this thesis will be to discuss how the logic of such progress comes out in particular mathematics text books, but the analysis may apply to groups of teachers just as well.

3) Does one find a Common Logical Sequence to what is taught, no matter through which approach it is taught?

Before this question is answered, one must first consider whether or not any criteria have arisen in this perspective, as occurred in the earlier ones, that indicate that the achievement of a given mathematical objective logically presupposes knowledge of other elements of mathematics. The argument thus far has been that it would make no sense for a teacher to ask his pupils to come to his class, until he had decided what is presupposed in learning to solve a linear equation, '$x + m = n$', say. However, the answer so far given in earlier chapters is that there is _something_ presupposed, and that this is agreed to by all perspectives. A more critical question is, '_Is the set of presuppositions for a given objective logically identical for all approaches_?' The answer to this would seem simply to be negative. One teacher may consider axioms and a definition sufficient background for teaching linear equations to a level at which the best pupils understand,

$$\begin{aligned} & x + m = n \\ \Rightarrow \quad & x + m - m = n - m \\ \Rightarrow \quad & x + (m - m) = n - m \\ \Rightarrow \quad & x + 0 = n - m \\ \Rightarrow \quad & x = n - m. \end{aligned}$$

Another teacher considers some understanding of Descartes' struggle to initiate modern algebra as a necessary presupposition. Here there may be differences of a philosophical and/or a pedagogic nature. The teachers agree that there are presuppositions involved in the given objective being attained, but accept no more than an overlap about what they are.

The atmosphere that this chapter has tried to generate, is one in which teachers would think twice before _always_ stopping at the axioms and definitions. It is by stopping there that they give the impression that, 'mathematicians start with an empty mind, set up their axioms and definitions at their pleasure, in the course of a playful free creativity... deduce theorems from these axioms and definitions' (_Proofs and Refutations_, p. 143, footnote 1). The argument of the science perspective is that this gives pupils a false impression of mathematical progress, if not of mathematics itself.

Thus, a provisional answer to question 3) is that a common syllabus cannot be guaranteed by simply agreeing on mathematical objectives. These objectives do not fully determine what precedes them in mathematical content, without regard to perspectives and philosophical movements. A more detailed response can be attempted once the fourth perspective has been presented.

CONCLUSION

At various points, this chapter has indicated the limitations that occur if one does not hold one hundred percent to the view that mathematics involves logically the method of science. This will be considered also in the concluding remarks that follow.

1. The result of ignoring 'mathematical discovery' in schools is that pupils come to believe falsely that they are the only ones who do mathematics 'untidily'. Even if one holds that 'mathematics' is 'tidy', one ought not to mislead pupils into believing that its development is equally formalised.
2. What pupils do, does not have to be called '<u>mathematical</u> discovery', if the teacher believes that it falls short of 'mathematics'. Equally unsatisfactory is for the teacher to call it 'discovery', when the pupil would happily accept it as 'replication'. However, the teacher ought to keep an open-mind to the possibility of discoveries, for otherwise they will be 'barred'.
3. The two methods of science presented may be used as models where they are incompatible with a philosophical movement. Thus the Newtonian View may be employed as a pedagogic model even by Hypothesisors.
4. There is one important difference between someone who employs the methods of science in mathematics teaching only for pedagogic reasons and someone who uses it, at least in a part of mathematics, on logical convictions. The latter person has a logical obligation to show that an appreciation of 'formal proofs' is an inadequate demonstration of being mathematically educated, if it omits an appreciation of 'mathematical discovery' methods.

C H A P T E R 1 0

MATHEMATICS TEACHING - TECHNOLOGY PERSPECTIVE

INTRODUCTION

Many of the points raised in the previous chapter have a bearing on the material presented in this one. To orientate mathematics teaching towards 'technology' is to presume some reference to 'science', or more particularly, to 'scientific methods'. 'Technology' is generally taken to imply the application of sciences to problems of everyday life. These sciences may be either 'natural' or 'social'. Business Studies is as much a technology as Engineering. In both subjects one finds mathematics joining with science to resolve problems related to everyday life. Thus, the 'technology perspective' focuses on the function that mathematics has, rather than on the nature or quality of the growth of the subject, mathematics. As with the 'everyday' notion of technology, it is assumed here that any product of a technology will have a use. The extent to which 'the technological perspective' demands that mathematics should consist totally of 'useful parts', is a central point of discussion in this chapter. The term 'useful' will be allowed a breadth of meaning to include 'being useful <u>in</u> mathematics', as well as 'being useful to some area <u>outside</u> mathematics'. Hence, those following a technological perspective are forced to ask questions about the purpose of mathematics, rather than presuming that it is a permanent member of the sacred timetable of any school. The most radical position is one which would claim that all mathematics has a useful purpose, and this should ground its place in education.

Such radicalism may be halted immediately by the suggestion that the technology perspective necessarily leads to a loss of rigour in mathematics education. Applicability is achieved at the expense of knowledge of formal proofs. Not one, but three problems arise:

Problem 1 involves the tension between 'rigour' and 'applicability'. Adversaries of the technological perspective suggest that a more flexible attitude to the demand for applicability would reduce this tension, but only at great cost to the technological perspective, as it will be outlined in this chapter. In considering various solutions to the problem, one central theme of this thesis will be highlighted, for each solution must be tempered, in relation to the mathematics educators' views of 'rigour' and 'applicability'. It will be seen that these views are embedded in the philosophies of mathematics discussed in part 1 of this thesis. If a teacher's notions of 'rigour' and 'applicability' are embedded in different philosophies, then he should not be surprised if there is some resulting tension. This leads one to identify the more general problem that has been present throughout this part of the thesis:

Problem 2 involves the influence, that the compatibility or incompatibility of the philosophies of mathematics identified in part 1 has, on the kinds of teaching perspectives that one can, with logical consistency, support. The point of concern brought out in this chapter is that teachers have the feeling that such a problem exists, but because they cannot identify it clearly, their solutions are pragmatic. This leads to difficulty in carrying through one's intended policy for mathematics education, for the teacher has an inadequate understanding of his teaching situation:

Problem 3 involves teachers presenting the mechanical application of certain techniques to pseudo-real-life situations,[1] as a substitute for

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1. By 'pseudo-real-life situations' one means situations described in terms of real life objects, like houses, cars, bills, et al., but they are not related to each other realistically, or the pupil is not asked a realistic question about them. When will one need to know 'the planes of symmetry of a matchbox' or the amount that a [Contd. overleaf

genuine applications. This devalues the notion of application, and fails to provide pupils with anything more useful than additional examination passes.[1] Even where the applicative area has genuine technological, or at least vocational, value, as is found in Commercial Arithmetic or in Mathematical Methods and Computer Studies, too often 'education' is sacrificed for 'training', in order that sufficient routines are learned.[2]

Once the standpoints of the differing philosophical movements are given with regard to these three problems, and the technology perspective in general has been clearly presented, it will be possible to turn again to the three questions raised in part 2 Chapter 6.[3] Throughout this discussion it will be of critical importance to distinguish philosophical, from purely pedagogic support for the technology perspective. It is very easy to interpret a perspective totally reliant on pedagogic support, as having stimulated concern on the basis of transient values only.[4] No matter the stimulus for implementing this perspective, the concluding argument of the chapter is that concentration on mathematics as 'servile', is not pejorative. Pupils who constantly look out for uses of what they know, rather than 'digest, regurgitate for examinations, and forget', are

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Fn. 1, p. 181, contd.
girl actually paid for a bicycle, if she resold it for £9 and lost 19% in doing so. These examples were taken at random from Mathematics through Experience, Book 3.

1. 'Examination passes' have a use, but those in Applied Mathematics seem to have a particularly short-lived one.

2. 'Sufficient' would be determined by the vocational bodies to which the teacher looked, or sometimes by an examination syllabus.

3. The questions were: 1) What distinguishes perspective X from all the others? 2) Must one logically have come to appreciate one perspective before, as a pupil, one can appreciate some other perspective? 3) Does one find a common logical sequence to what is taught, no matter through which approach it is taught?

4. Rather as one considers a neighbour to be always 'keeping up with the Jones', there may have been a tendency in times of greater affluence for Heads of Department to compete for the most 'with-it' mathematics equipment or project, for expediencies that may alter with each change of Governors, say.

closer to most people's conception of a mathematically educated person, than at first sight might be expected.

APPLICABILITY is not a synonym of technology. It has already been mentioned that technology is taken to refer to 'science', while 'applicability' is a broader term, equally at home in the arts or the sciences.[1] Furthermore, the technological perspective, as discussed here, includes applications of mathematics within mathematics, as well as outside mathematics. This is not excluded by the concept of 'applicability', but is not usually considered. One may use a spanner to open another spanner, but one would not take this to be its central use. While most of the debate is about 'external applicability', and by 'applicability' one will generally be referring to 'external applicability', it ought to be remembered that this internal reference is acceptable, and will be mentioned explicitly at times. 'Internal applicability' takes a back seat, because far less heat is generated by the suggestion that mathematics must be taught through its 'internal' uses, than when one suggests it must be taught through its 'external' uses too. Clearly, at least two forms of the technology perspective are possible. These positions are:

a) a Strong Technology Perspective insists that mathematics is to be seen as necessarily applicative. Every element of mathematics awaits service elsewhere. All mathematics has the potential for application, from the moment of its invention.

In the last chapter Sawyer was quoted as saying that a necessary part of establishing a mathematical theorem was that, 'fruitful applications of it have to be found' (see p. 166 above. This is a stronger position even, than that found in part 1 Chapter 4. There, mathematics was identified as 'the science of possibilities', not the science of

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1. A very broad sense of 'science' was mentioned on p. 157 above, which roughly equated 'science' with 'knowledge' and thereby could cover 'the arts' as well as 'science'.

'fulfilled possibilities'. Nevertheless the logical point is held by both: that mathematics consists totally of elements necessarily capable of 'fruitful applications'.[1] As Morgan notes of an examiner for the International Baccalaureate in Mathematics, he 'is required to assess the student's...ability to represent situations in mathematical terms (mathematical models), to examine their implications and possibilities, and to arrive at definite conclusions by the application of mathematics as a tool' ('The International Baccalaureate' in Developments in Mathematical Education, p. 259). This is certainly on lines similar to Sawyer, and it is found within the confines of an examination structure.

To put this notion of the Technology Perspective more concisely, 'Mathematics or some part of it is to be seen as consisting of elements all of which necessarily have the potential to be employed in the resolution of problems in other areas, either inside or outside mathematics.'

In this strong conception of mathematics as technological, the phrase 'inside or outside' is of critical importance and deserves elucidation. To the Hypothesisor, mathematics stops being definable as mathematics if it lacks interpretation. Furthermore, interpretation must logically occur outside mathematics, for mathematics to take on the character of a discrete discipline. An economist may be given the qualifying characterisation 'mathematical economist', precisely because he has a significant knowledge of another discrete area; i.e. mathematics as well as economics. From the basic problem of simple interest, to complex models of the economy described within computer packages, the economist is aware that arithmetic formulae are open to useful

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1. Sawyer's stronger position amounts to barring theorems from full status until they are fruitful, while the Hypothesisors argue that all theorems are potentially fruitful, and the more explicitly fruitful parts are the parts that the present community remembers.

interpretations. In arithmetic, $I = P \times t \times \frac{r}{100}$ just indicates an acceptable way of manipulating real numbers, while it is interpreted by the economist as an indication of the nature of simple interest. Its status as a well-formed formula of arithmetic is presumed by the economist. He does not check that this is so, but assumes that members of the related area, in this case, mathematics, have confirmed its validity. However, the economist is confirming the existence of the discrete area, known as mathematics, by taking $I = P \times t \times \frac{r}{100}$ as validated elsewhere. If there were no members of other areas who were willing to make such public confirmations of trust, then mathematics would be quickly reduced to the status of a game. It is the fact that mathematics is interpreted 'outside' mathematics, that is crucial to its being regarded publicly as a discrete area of knowledge.

b) <u>a Weak Technology Perspective</u> insists that mathematics has the potential of being helpful sometimes, to other areas, either inside or outside mathematics, through its application. The argument is that knowledge and understanding of certain parts of mathematics are found to illuminate empirical situations, if one has the relevant knowledge of both the empirical situation and mathematics. Someone holding this weak perspective may claim that the arithmetic formula mentioned earlier in the economics example, does find an interpretation in economics as the Simple Interest Formula, but it is an exaggeration to claim that any well-formed arithmetic formula necessarily has the potential for such fruitful application.

<u>External Applicability</u>. Whatever one's notion of the technological perspective, one is using the term 'external applicability', at least implicitly. In order to understand the possible diversity in this phrase, the approaches to answering these three questions will be considered:

1) 'Where should you build the greenhouse?'

2) 'How could you have made sure that that joint fitted better?'

3) 'Cut that bun in four and have a piece. What part of the bun have you eaten?'

1) This question, involves the many variables that there is no space to identify and discuss, but somehow a decisive criterion has to be identified by which to judge the best place to put the greenhouse. Such a criterion could be, 'sunshine hours[1] on a maximum area of the greenhouse'. Here practical experimentation is <u>not</u> practical, for one cannot easily go around erecting and re-erecting a greenhouse until the optimum site is found. The question cries out for a model to illuminate the situation. Aided by trigonometry and calculus, a mathematical model will be derived. It will not solve the problem on its own, but then as Professor Lighthill says, 'no one can expect to solve the <u>whole</u> of any problem mathematically' ('Presidential Address' in <u>Developments in Mathematical Education</u>, p. 98). Here is a problem and someone with a kit of mathetical models is likely to find one that provides 'a good fit', perhaps with one or two on-the-spot adjustments, in line with a physical model, no doubt.

2) This question involves a previously constructed object, for example a cot, and the amateur carpenter is not satisfied fully with his achievement. He did it all by 'eye', a straight edge and a pencil, rather than according to a previously designed plan. He now sees that as a matter of fact he would have done better to have used some simple geometry, before he had plunged into estimations.

3) This question arises when an exasperated teacher combines a number of motivational techniques to overcome a particular child's inability to handle simple fractions. The question is not just concrete, but

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1. This is to include both a measure of intensity and time.

involves both an experiment and an unsolicited reward. The teacher is not interested in what the pupil learns about currant buns, but only what he learns about fractions.

Looking at the three questions, one might suggest that the three questions are essentially the same, for they all employ empirical situations and some mathematics. However, the argument of this chapter is that each identifies a distinctive use of the term, 'external applicability'. These will be called, Rational Applicability, Concrete Applicability, and Pseudo-Applicability.[1]

1) Rational Applicability is to be employed when there is no reasonable alternative for dealing with the problem, except through some abstract characterisation of the problem. It is not realistic to imagine greenhouses being built and dismantled all over ten acres of land, for a number of years, until the problem is resolved. One can conceive of this happening, but one would consider it irrational to turn the conception into actual practice.

2) Concrete Applicability is worth separating from rational applicability, because a teacher who is given examples of the kind expounded under 'the amateur carpenter problem' only, may believe wrongly that mathematical applicability is motivating, but not rationally compelling. Thus the phrase 'concrete applicability' is to be employed to refer to problems where it would be reasonable to choose either to use a theoretical framework to resolve the problem, or to rely largely upon practice. One alternative is not overridingly the more reasonable, as would seem to be the case with the greenhouses and the models of national economies. The carpenter could reasonably choose to construct alternative joints, rather than calculate which shape will fit most tightly, but

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1. 'Pseudo-applicability' will be treated separately in this chapter, for it is only meaningful in the restricted domain of mathematics education, and not in ordinary usage.

this too would be a reasonable approach.

Both these forms of applicability focus on realistic problems, whose solution is of interest in both the area applied and the recipient area.

CRITIQUE OF PSEUDO-APPLICABILITY

Pseudo-applicability only occurs in educational situations. It involves contrived empirical events, being employed to help eliminate ignorance in a given area of mathematics, like fractions. For motivational reasons, all or some part of mathematics is presented to a pupil through empirical situations. Thus Thwaites argues that underpinning S.M.P. is the 'philosophy of drawing mathematics out of real situations which are easily recognizable, readily comprehensible and genuinely interesting to the pupils' (S.M.P., the first ten years, p. 176). Using this approach initially does not necessarily prevent graduation in a given area of mathematics to one of the other forms of applicability. A concept acquired by abstraction from empirical situations may be employed on another occasion to resolve problems in new empirical situations. Tommy may learn what a quarter is, through currant buns, but tomorrow he may be sharing out chocolate with three of his friends. This seems to indicate Thwaites' intention, when he goes on in the same report, 'The familiar "Oh Sir, what's the point of this?" should now be as obsolete as totting should have been a generation ago. Thus every chapter, especially in the first few books, starts with a practical task which is aimed at stimulating the pupil's imagination and sharpening his curiosity.' However, the practical tasks begin chapters rather than end them, and even in books like Book F where much is made of the four applications of matrices, all are internal to the same parental structure, algebra.

The point is that one may indicate by mathematics in application, illumination primarily of mathematics itself, rather than the empirical area to which it is supposedly applied. Consider the following problems from S.M.P. in this light, 'Make a list of at least 10 triangular objects or frameworks that you have seen' (Book A, p. 44). This seems to reinforce triangularity rather than improve one's understanding of the world. OR 'A gun weighing 3 tons fires a 25 lb. shell with a muzzle velocity of 2000 ft. per sec. What other information would you require to calculate the recoil velocity of the gun?' ('A' level S.M.P. question for 1964). This stimulates thoughts of standard dynamics models rather than illuminating ballistics.

In pseudo-applicability, it does not matter what clothes are put on the mathematics, for only mathematics is identified in the educational objective.

Applicability of mathematics as the objective of training. Even though the status of mathematical knowledge has been questioned, there remain features of mathematics that make it attractive to other areas of study. These are the features that are commonly agreed as necessary conditions of mathematical proof. The possibility of sufficiency may be removed but there is still agreement across the philosophical movements that objective, public formulation for example, is a necessary condition of mathematical proof. Another feature is the ease with which mathematical procedures may be repeated.[1] The attractiveness of mathematics to the social sciences, in particular, may be tainted, but is not lost.

The effect of this in schools is a practical concern for pupils to have 'enough mathematics' to meet the requirements that are to be found in the sciences. Various kinds of 'applied mathematics' have arisen; sometimes under this title, and sometimes under titles suitably adapted to the market for which it was intended: for example, 'commercial

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1. This reiterates the point made on p. 169 above that mathematics is distinct, for it is never concerned with particulars but rather with 'classes', 'sets' or 'types'.

arithmetic' for future secretaries, and 'mathematical methods and computer studies' for future computer engineers. Consider the example of third years in a secondary school, learning how to use a table of squares; e.g. $13^2 = 169$.... In one class they learn to practise this skill through the problem of carpeting a six bedroom mansion, all of whose rooms are square. In another class, the previous lessons have included discussions on the value of learning one's 'Times Tables', and how prewritten tables may reduce labour, and also on patterns to be found in the sequence of square numbers. The pupils in this class therefore, now 'find' it useful to employ 'Square Tables' to facilitate obtaining the objective of their present work: the gaining of an approximate value for gravity through experiments using Newton's trolley.

This is a crudely presented contrast of styles but it does pick out a genuine distinction. Only in the second class have the children freedom[1] to suggest firstly that 'Square Tables' would be of use, and later, that 'Square Root Tables' would be useful also, to reduce their labours. In the first class the children can competently employ an algorithm, but only in the second class are they given the opportunity to demonstrate that they recognise differing situations in which a particular algorithm can be appropriately employed. No one wishes to reject the value of combining applicative and vocational approaches with 'general education', provided the result is education and not training. Sometimes the vocational requirement is the sole objective, and the result is the machine-like application of standard methods and models, to standard problems, rather than understanding the criteria of

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1. 'Freedom' is not unlimited and a phrase like 'guided rediscovery' may be appropriate given the discussion in the previous chapter on this point. Ennever uses the clause, 'unconscious of any restrictions they have accepted in making their own selection' (See, 'Science in the Middle Years' in Schools Council Working Paper, No. 22, pp. 60-1). The teacher may attract pupils to a given decision, but does not force it upon them.

application. Few people today expect to see 'teachers from Training Colleges teaching Religious Instruction and Physical Training' but rather, they expect 'teachers from Colleges of Higher Education teaching Religious Education and Physical Education'.

Unfortunately the inclusion of the name of an accepted discipline in the title of a course may leave it uninspected for educational content. Children may be left, therefore, passively to apply standard algorithms, and mechanically resolve artificial problems, or even real ones, that could be more efficiently resolved by some other means. Thus, one does not need to use a computer to sum the first 100 integers, when one can use a formula. Realistic use of mathematics is essential if pupils are not quickly to become disillusioned.[1] If there are such 'training courses', then they can become 'educational', only if the pupils are encouraged to acquire knowledge and understanding of both mathematics and the areas in which it is to be used. Even on motivational grounds, one would expect this to lead to greater concern by pupils for what they learn, if they appreciate the purpose of their applicative work. Fourth years would then learn how engineers <u>really</u> use mechanics, in reducing frictional resistances of aircraft wings, rather than how to deal with the problem of a friction-limited projectile. If one has no understanding of either the mechanics, or the physical properties of aircraft movement then one cannot develop one's 'applicability education', and there is certainly room for the argument that this rules out the possibility of mathematics education for all.

APPLICABILITY AND MATHEMATICAL RIGOUR

There is a fear among some mathematics teachers that the price of 'external applicability' is 'rigour'. Underlying this is the belief

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1. This point was noted earlier, p. 163, footnote 1, with reference to 'discovery' in science teaching.

that mathematics consists of a priori knowledge, while science consists of a posteriori knowledge. Thus if one concentrates on technology, then one is working solely in the a posteriori area. This sound logic focuses hard down on the following:

1) How do non-mathematical areas employ mathematics? Lighthill asserts that 'effective application is possible only if one sets out to learn the language of the field of application and master those characteristics special to it' (loc. cit., p. 96). Lighthill suggests a constant interplay between mathematics and the field of application. It is logically possible for mathematics to be employed in science, without the mathematician who works for the scientist understanding the scientific problem. He is simply asked to resolve a given partial differential equation, but knows no more of the situation than that. The scientist must, in Lighthill's terms, be capable of 'building a bridge between the abstract ideas and inferences of mathematics and the concrete problems arising in some field of application' (p. 95). If the mathematician relies totally on the scientist, then the assumption must be that the mathematics is separable, isolatable from the empirical. There is a backing for this separation in the identifying of scientific a posteriori knowledge and the identifying of mathematical a priori knowledge. This position is presented, for example by Koerner in Fundamental Questions in Philosophy (p. 85), and earlier but more fully in his Experience and Theory (Chapter XII).[1] This activity by a scientist with a problem, can be encapsulated in the term 'idealisation'. An idealisation is the process by which a model is produced of some empirical situation. The process

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1. This is also the position held by the applied mathematician, Hall, who says of applied mathematicians and scientists: 'there is no necessity for each to do the work of the other' (See 'Applied Mathematics' in Mathematical Education, p. 32). This is the call of 'co-operation' and may be contrasted with Lighthill's desire for 'bi-lingual' applied mathematicians.

is necessarily selective, in the sense that not all features of the empirical situation are represented in the model. That is, 'no two empirical propositions...can be identical with their respective theoretical idealizations' (Fundamental Questions in Philosophy, p. 84). The idealisation possesses all relevant features with regard to the problem at hand, and meeting this requirement is no easy task.

As Lighthill points out, for the representation to be sufficiently true of the empirical situation the 'applier of mathematics must learn... to use simultaneously the weapons of mathematical reasoning and the inferential methods typical of the field of application' (loc. cit., p. 96). This process of constant interaction is another form of 'application'. It is distinguishable from 'idealisation' because 'interaction' is that process in which 'applicans' and 'applicandum'[1] modify each other, while in idealisation the process has only one direction.

Thus two methods of applicability have been identified. These are 'idealisation' and 'inter-action'. Philosophically idealisation is compelling for logicists. This method indicates a separation of 'applicans' and 'applicandum', and furthermore, the a priori knowledge can modify the a posteriori, but not vice versa. This fits the logicist system, while inter-action would require the possibility of change in the opposite direction, which is not logically acceptable. Intuitionists retain a modified choice between the alternatives, for it is important to remember that public interpretations of mathematics in Intuitionistic logic are synthetic a posteriori, and so could conceivably be open to modification from outside mathematics. However mathematics in itself

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1. 'Applicans' is the instrument of clarification: the mathematical modelling, and the 'applicandum', the empirical or other situation, which is identified as problematic. 'Interactive application' will be used to indicate that 'applicans' and 'applicandum' are mutually modified through modelling, on occasions.

would not be modified. Even without the logical constraint the Hilbertian Formalists[1] would argue that as a matter of stipulation mathematics consists of formal systems, empirically founded but deductively expounded. Thus inter-action is stipulatively excluded, if one means that mathematics itself is open to modification, and not just limited to the appropriateness of the model. There is an ambiguity in the line, 'We then manipulate the model in such a way as to mimic certain features of the proposed development we are investigating' (Teaching Mathematics Applicable - Introductory Guide, p. 12). This hypothetical model is formed in mathematics, taking constant cognisance of both mathematics and the area of application. The hypothesisors can allow logically the possibility that inter-action will lead to fundamental changes in mathematics itself,[2] but more usually one is changing one previously known mathematical kit for another, under the influence of the 'applicandum', the empirical situation.

2) Idealisation and Mathematical Rigour. The argument of 'idealisation' is that applying mathematics involves the permanent retention of the control of mathematics by mathematicians. In being asked to provide a model for a given physical situation, the door to the mathematics room remains shut to the scientist and his influence. The scientist is to present his problem in such terms that the empirical features fall away like the skin of a ripe peach, leaving the exact concepts of mathematics naked, behind.

Thus, in the simplest of examples, the problem of the addition of apples, Koerner's point would be that the solution to any such problem

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1. Strict Formalists sidestep this debate, as for them, mathematicians have no responsibility for the interpretation of formal systems, e.g. in applicability.

2. Wilder discusses this point in Evolution of Mathematical Concepts, pp. 11-16 and concludes, 'mathematics...evolves under the influence of forces...both within itself and without'.

must be a posteriori knowledge.[1] This assumes that, 1) any proposition that describes the world is part of a posteriori knowledge and 2) any proposition of mathematics, if it has a subject matter, is logically different from that of empirical propositions. Thus '1 + 1 = 2' does not describe the world, but is a formal model of occurrences of empirical descriptions, like '1 apple and 1 apple make 2 apples'. The former proposition does not exactly identify a translation of the latter proposition into mathematics, but is only an idealisation of such.

Given this strict separation of mathematics from the empirical contexts, any lessening of mathematical rigour cannot be logically tied to applicability, if it occurs. Mathematical propositions do not occur[2] in empirical discourse itself. Thus the mathematical propositions may be underpinned by any one of the philosophical movements identified in part 1 of the thesis, with equal facility.

3) Inter-action and Mathematical Rigour. The argument in support of 'inter-action' is that applying mathematics involves the continual modification of both mathematical models and scientific theories, according to empirical evidence in the former case, and to computations and proofs in the latter. A model that 'sends' a space rocket into the sea requires modification, no matter how beautiful the proof is. The repercussions may not be so visible in reality, and so the mathematician merely receives a report of the experiment say, that was carried out at the end of a chain, of which his mathematical model is an integral part. Only the 'bi-lingual' will appreciate to what degree the empirical difficulties are the result of an inadequate mathematical model. This is particularly

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1. See Experience and Theory, pp. 167-72 and 'On the Foundations of Mathematics in Experience' in L'Age de la Science, Vol. III, No. 3.

2. What occur in empirical discourse are names of mathematical propositions. Thus '2 + 2 = 4' may occur but not, 2 + 2 = 4, or more likely one finds the 'cross-breed' 2( ) + 2( ) = 4( ) in which the exact mathematics and inexact scientific statements are conflated.

the case, if the model has been fed into the scientific theory at several points, rather as statistics may be found in economic forecasting.

This position only makes sense, if empirical features can be said logically to modify mathematical elements _and_ vice versa. In order for this to occur in a logical sense, simultaneously, both the empirical and the mathematical knowledge inputs must be either _a priori_ or _a posteriori_, and not one of each. Given the Newtonian view outlined in the previous chapter (see p. 159 above), it may be possible to make all theoretical components _a priori_, but it is more likely that they will be considered _a posteriori_. In this case the logicist would feel logically excluded, just as hypothesisors would feel excluded if the _a priori_ position were taken throughout. As the mathematical components might be those of intuitionistic logic, no other movement besides the logicist would feel excluded necessarily by _a posteriori_ knowledge throughout. Thus the logicist would say that this 'Evolutionary[1] inter-action' has less rigour than mathematics itself, and must be employing an empirical substitute for mathematics. The logicist may yet accept this same inter-action method for applications within[2] mathematics, while rejecting it for external use.

The most obvious examples of this 'Evolutionary inter-action' would be at the rather advanced level of Hilbert spaces applied to quantum mechanics. Yet it is possible to involve interaction at the secondary school level, if the problems have sufficient realism. This is just one of the principles of 'projective modelling' that is the key to the design of the Schools Council Sixth Form Mathematics Project. Students are constantly involved in two activities, 'thinking realistically' and 'thinking mathematically', 'translating' and 'simulating'.[3] The interaction can be

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1. Using the terminology introduced above, part 2 Chapter 9, p. 160.
2. Like algebra applied to Euclidean geometry, or matrices to algebra.
3. See _Teaching Mathematics Applicable Introductory Guide_, particularly pp. 20-4.

found also in first year books, like Modern World Mathematics, where both mathematical and physical models are required and the differences discussed.[1] In this way the models are open to mutual modifications, and the pupils are encouraged to develop understanding in a genuinely inter-disciplinary fashion.

The inter-action is the criterion by which one decides that rigour is vital, and if there is no inter-active concern[2] at a given point, then the force for rigour is reduced. As Ormell puts it, 'One cannot expect students to appreciate excessive formal rigour on the new course.... The essence of rigour is taking care over inferences of various kinds....The purpose of rigour is also more evident: a mistake in inference in modelling might entail something going disastrously wrong in the real world.' (Teaching Mathematics Applicable Introductory Guide, p. 23). 'Rigour' is tied to 'caring' rather than exclusively to an obedience of axiomatic sequences. Nevertheless, in Ormell's philosophical stance, there is only support of the underlying logic upon which 'formal rigour' rests, rather than a constant requirement of its detailed scrutiny. Ormell follows Douglas Quadling in the belief that, 'to be mathematics at all, there must be the possibility of deriving it by processes of logical argument from a recognised axiomatic basis' ('Issues at Secondary Level', p. 178 in Mathematical Education).

There is no challenge here to classical logic, but clearly Intuitionists do explicitly make such a challenge,[3] and also among the

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1. See for example, p. 73, question 6 of Book 1 which involves a mathematical description of the movement of a curtain ring, and also the construction of a physical model of such movement. It draws out the point that the model produced for a taut material must be made more general - modified - if one learns experimentally that the material holding the curtain is elastic. This is Inter-action at work.

2. Indelicately, one might say that extra scrutiny occurs when one treads on another's toes.

3. See above p. 48, Heyting argues 'logic is a part of mathematics, and can by no means serve as a foundation for it' (Intuitionism, p. 6).

Hypothesisors the unique position of Classical logic is at least questioned.[1] 'Rigour' is a much less clear notion, once it is allowed that the ground rules may alter. In this way, 'rigour' becomes an ambiguous concept, if the 'formal rules' are not made explicit, but are mistakenly taken as read. As the discussions of part 1 of the thesis have shown, one cannot take 'mathematical knowledge' as unquestionably absolute, any more than one would take a scientific theory as believed by all and for all time. Once one accepts that mathematics has no absolute status derived from an eternal logic, for a considerable number of people concerned with mathematics - say, for scientists or philosophers as well as mathematicians - then it is less of a surprise that its justification as an essential feature of what any man must know, should be held by many to be its 'usefulness', rather than its 'peculiar logical form'. In this light, attention is drawn again to the conclusions reached on p. 89 above, that 'derivability' is a sufficient condition of 'proof' for Logicists and Formalists, and for the public form of Intuitionistic proof, but not for the Hypothesisors. They require in addition, the 'public scrutiny of proof-analysis' and that the proof has 'applicative value[2] either within or without mathematics'.

Thus three points need underlining with regard to the nature of rigour:

1) 'Rigour' is related to 'derivability'. By 'derivability' one means, 'derivability within a given logic', usually either Classical or Intuitionistic. This in itself indicates a plurality of rigour.

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1. See the argument in part 1 Chapter 4 which concludes on p.68, footnote 1, by noting that Lakatos' editors fear any suggestion that there are no infallible methods of proof.

2. Compare p. 166 above, W. W. Sawyer's comment on the final stage in the discovery of a mathematical theorem, 'Fruitful applications of it have to be found'. The point implicit in Sawyer and explicit in Wilder is that mathematical knowledge is 'public', and a mathematical theorem is not what one man produces in private, but what a community publicly recognises as having a 'relation to the world of reality'. See p. 164 above.

2) There is no complete agreement on the importance of rigour in mathematics. The Hypothesisors stress the co-existence of 'rigour' and 'inter-action'.

3) For the Hypothesisors any adequate analysis of 'rigour' must include its purpose, in addition to criteria of use, like 'setting out the formal rules and checking that a given derivation follows these rules' (G. Kreisel, 'Informal Rigour and Completeness Proofs' in The Philosophy of Mathematics, p. 78). Point 3) is a logically stronger form of point 2).

Given the validity of point 1), a teacher ought to be aware that 'rigour' must be ultimately linked to a given logic. The teacher may then take either the position that there is only one true logic, the position of 'absolute rigour', or that there is a plurality of 'rigour', in the sense that rigour is identified relative to a given logic. Further, points 2) and 3), if either is accepted by a teacher, imply that 'checking...a given derivation' is more important on some occasions than on others. In other words, rigour is to increase in strict co-relation to inter-action. This may be a valid attitude to instil in pupils, but it is easy to see the dangers of taking it as freedom to be lax, on most occasions.

One central facet of the argument has been omitted, and will remain so until the beginning of part 3. That is, the conception of mathematics education and more generally, of 'education' itself, that is held in this thesis. This will be remedied then, and the aims of mathematics education will be identified at the same time. In what follows, it will be assumed that the technological approach does aim to get pupils to understand both what they are doing and why they are doing it. This eliminates most common uses of 'training'.[1] It is not inconsistent with the notion of

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1. The concern for 'understanding' rules out the possibility of the pupils 'being just trained' (R. S. Peters, 'Aims of Education', pp. 19 and 54 in The Philosophy of Education). Gribble emphasises the place of 'explanation' in education and its omission from training (Introduction to Philosophy of Education, pp. 22-4).

'rigour' identified by Logicists, Formalists and Intuitionists for people to be trained to be rigorous, for one may learn to take great care in following a logical sequence, without any appreciation of why one is taking so much care. Mathematicians tend towards rigour when there is greater likelihood of error, but pupils may be trained to be rigorous without being given any explanation, except for the teacher saying that it is an important requirement for examination success.

THREE QUESTIONS

Having clarified the nature of the technology perspective, it is possible to raise the questions first identified in Chapter 6; and proposed on p. 182 in this chapter.

1) What distinguishes the Technology Perspective from all the others? According to the strong position, all mathematics taught by this approach is taught as potentially useful.[1] This usefulness may be satisfied in two ways. Either the mathematics is useful within[2] mathematics or to some other area. According to the weak position, the usefulness may only be contingent. Thus, a teacher has two distinctive approaches from which to choose:

a) A Strong Technology Perspective in which mathematics necessarily contains both internal and external applicability.

b) A Weak Technology Perspective in which mathematics is contingently externally applicable.

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1. A point hidden in the present argument is that while mathematics may be defined as a set of potential models, not every application of an element of mathematics is modelling. All modelling is applicative but within mathematics, not all applying is modelling. One may have proved 'the angles on the same segment theorem' and then one uses it, to show its fruitfulness, to identify the nature of cyclic quadrilaterals. There is applicability without modelling here. Yet if one uses algebra to demonstrate a property of arithmetic then algebra models arithmetic to achieve this. For example, to demonstrate commutivity of integers under addition.

2. In which case, one could treat mathematics as if it were a technology, rather than as a technology. See p. 196 above for a logicist justification of such a position.

This is not the end of the matter, for one may argue that mathematics is internally, nothing but a set of potential models, and also argue that the use of these models externally, is a contingent possibility. This may be identified as 'A Strong Internal Applicability Perspective'. It contains logical reasons for presenting some of mathematics as internally applicable. This perspective may be viewed as no more than a modified form of b) for it does not deny b). However, it is useful to give the perspective a separate identity, because it has obvious affinity with those schools of thought that are most attracted to the formalisation of mathematics; including Logicists, Formalists and Bourbaki.

However, the Logicist philosopher presents a view of internal applicability that has not been inferred at all, thus far. The logicist identifies a special relationship between propositional logic[1] and mathematics, in which logic models all mathematics. That is, any element of mathematics may be redescribed in principle, in terms of propositional logic. The logicist is concerned solely with the applicability of logic to substantiate the rigorous nature of mathematics. In the sense of model that was identified earlier,[2] it is inappropriate to suggest logic models mathematics for there is no possibility of inaccuracy in the explanatory role, as the logicist identifies it. However, applicability has been differentiated from modelling, and so 'internal applicability' is a meaningful interpretation of the logicist position, as there is necessarily the interrelationship of all elements of mathematics, through their reduction to logic (or set theory, in Bourbaki and Quine). Thus it is within a logically rigorous framework that S.M.P. Book F introduces matrices, in order to solve simultaneous equations, and Modern World

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1. More realistically one might refer to the Bourbaki redescriptions of mathematics in terms of set theory.

2. See p. 159, footnote 1, where a model is identified as a useful but not a true description of something.

Mathematics Book 1 introduces linear equations to understand translations and co-ordinate sets. The logicist puts 'logical strength' above the fruitfulness that is so important to the Hypothesisors, in particular. As far as the logicists are concerned, applicability has no intrinsic value but occurs only for the sake of improved rigour. In this sense, the logicist is interested in internal applicability for the logical support it gives mathematics, but this cannot be the case with external applicability. As the logicist views logic and mathematics as a discrete area of knowledge, consisting of analytic a priori propositions, then this area can only be contingently connected with any area consisting of a posteriori propositions, which includes all science.

The Formalist may either consider the strong technology perspective as meaningless, or will have a view of internal applicability much along the lines of the logicists or Bourbaki, that there is necessarily the possibility of interrelationships, given the nature of the rules of manipulation. He would therefore accept the Intermediate Perspective, and certainly this would seem highly compatible with Formalism as described in this thesis. On the other hand, external applicability does not follow necessarily as a consequence of any definition of formalism that has been presented, nor is it incompatible with any. As was the case with logicism, formalism excludes the Strong Perspective, for formalism identifies mathematics as essentially interpretationless. If mathematics was necessarily externally applicable then this would feature in the meaning of mathematics, and would prevent its being an essentially interpretationless formal system. It has been strongly suggested in the discussion of the Games perspective on pp. 150-54 above, that external applicability can only be contingently linked to the Formalist conception of mathematics. External applicability was recognised in that discussion as a facilitator of mathematical understanding at pre-formal conceptual levels, and so it is those formalists who are most concerned to preserve the view that mathematics is really uninterpreted, that must entertain the technology perspective for pedagogic reasons alone. While formalists may be particularly suited to the Intermediate Perspective, it is also compatible with Intuitionism and the Hypothesisors.

The Hypothesisor who defines mathematics as the science of possibilities, is implicitly identifying mathematics with 'the science of applicability'. Mathematics is identified as a system of potential models and modelling is a 'strong' form of applicability. There is thus a logical connection between applicability and this philosophical movement. Furthermore, in the works of Ormell, Sawyer, and Wilder one finds a logical commitment to both external and internal applicability. This philosophical movement does not require a logical distinction to be drawn between 'internal' and 'external' applicability, and so the connection between mathematics and applicability is between all elements of mathematics being either internally or externally applicable, or both. Thus the commitment is to the Strong Technology Perspective.

No matter what philosophy underpins one's commitment to the technological perspective, 'applicability' characterises its distinctiveness from the previously identified perspectives.

2) Must one logically have come to appreciate one perspective as a pupil, before one can appreciate some other perspective?

Given three forms to this perspective, it is clear that the Weak Technology Perspective makes no necessary claims on the logical priority of approaches, beyond those that will be identified in discussing the other forms, for it amounts to saying that a teacher has a free hand to motivate pupils however he wishes. The Intermediate Technology Perspective introduces the important question of the logical status of internal applicability as being separable from that of external applicability, but the 'science of internal applicability' is only a technology in an extenuated sense.[1] However, the focus of the previous chapter has been methods of scientific investigation rather than the identifying of mathematics as a science. Strictly speaking, an understanding of such a form of investigation is presumed when a teacher wishes his class to see that 'algebra' is in the role of a technology, when it is used to illuminate 'Euclidean geometry'. The teacher ought to be claiming that algebra provides the tools for scientifically investigating geometry, in just that sense of 'scientific' identified for 'the Newtonian view' on pp. 159-60 of the previous chapter.[2] In this sense, to appreciate that geometry can be 'algebratised' is to understand a scientific method, and hence that one area of mathematics can be applied to another. This is not to claim that the teacher must separate the presentation of the scientific and technological perspectives in time, but only that understanding of the latter includes understanding of the former.

It was suggested in reply to the first question that indifference to philosophical standpoints does not carry over to the Strong Perspective. The implications of logical force there, identifies certain necessary characteristics of mathematics, in particular, that it is a science that involves hypothesising - answering 'what happens if' questions. Hence, the mathematically educated child must be able at least, to follow a method of science and hypothesise, even if he is not required to understand the method. In the previous chapter two methods of science were picked out and one of these centred on 'hypothesising' and was seen readily to fit one of the philosophical movements, the Hypothesisors. There was not however, a uniqueness claim made. Thus pupils will be

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1. Just as it was noted at the beginning of Chapter 9 that 'science' has an extenuated sense that covers all search for knowledge, hence one could see 'technology' as covering all instances of 'applicability'. It would be however only technology in this sense, and not according to common usage.

2. There is the inductive generalisation that geometry can be fitted into an algebraic framework, and that resulting proofs match original Euclidean ones for power (see p. 105, footnote 1).

required to show that they can follow a method but the preferred method is open to modification by 'the community of mathematicians'.

One peculiarity of the method identified by the Hypothesisors is that no distinction is drawn between the method of mathematics and that of mathematical development. Mathematical proof and mathematical discovery are inseparable for the Hypothesisors. Development is not to be hidden (see the earlier discussion pp. 62-71 above). It may be as a result of this desire to identify mathematics with what is clearly and cleanly demonstrable that 'idealisation' arose. This fits much more readily with 'what happened when' questions than 'what happens if'. To face uncertainty would be a radical departure for many mathematicians and mathematics teachers but that is what the Hypothesisors and the 'inter-action' method require. Teachers who believe that motivation is engendered in otherwise bored pupils by this approach, may turn to a series like Daily Life Mathematics in whose preface one reads,

> It is a common experience of teachers that pupils put forth their greatest efforts when engaged on work which they realise is worthwhile, and further that the practical approach and visual methods of teaching are the most effective means of arousing their interest...the pupils should have an opportunity of learning something of their place in the universe...topics may be grouped under two main headings, those concerned with financial transactions and those concerned with space and time.

An important criterion employed in syllabus selection in this book is to ask 'in what circumstances, in daily life, does one ever need to...' (Preface, p. vii).

In this series the 'daily life' approach is not chosen for motivational reasons alone, for there is likely to be far more mathematics that could be taught through this approach than is actually included. Criteria for selection of what is to be taught must go alongside how it is to be taught. A teacher who is just concerned with motivating his pupils, might use a pin to select material. However an orientation towards 'education' and not simply, 'the learning of mathematics', is likely to involve additional criteria. One finds in the above extract from Daily Life Mathematics a possible conception of education, in terms of 'learning something of their place in the universe...'. This would certainly narrow the selection of material for it is to look both to the pupil and his future citizenship, for 'relevance'. This can be carried out free of any other perspective. The teaching involves indicating how the mathematics chosen will be worthwhile in the life of the pupil, but there is no obvious attempt to present or exclude any one philosophical standpoint.

It would be possible for pupils to learn to hypothesise within mathematics, dealing with its internal hypothetical nature, without being given the challenge of applying hypothesising externally. This could be done even in line with Sawyer's criterion of 'fruitfulness'. Thus, if one does not hold the life-mathematics condition within the Hypothetical philosophy then one may argue that the science perspective identified in the previous chapter, provides all that the technological perspective requires with logical force. What is left disputable is the notion of 'science' itself and whether it is as the formalists employ it, possibly interpretationless, or as indicated on p. 158 above, as what helps 'man to deal with his environment'. If the latter definition is accepted then the science perspective is an empty vessel, an instrument awaiting direction. To return to the question, the conclusion is that the technological perspective does presume an appreciation of the science perspective, and so a pupil must have come to appreciate that perspective in order that he can appreciate the technology one.

3) Does one find a Common Logical Sequence to what is taught, no matter through which approach it is taught?

Largely one may reiterate the answer given in the previous chapter on pp. 177-9. That is, mathematical objectives concerned with the content of what is to be taught will necessarily identify certain prerequisites, but not all. Agreement that all thirteen year-olds should have knowledge and understanding of a rigorous proof of Pythagoras' theorem will entail necessarily prior knowledge of right-angled triangles and so on, but differing teachers will present one or more differing proofs, and only a teacher concerned with 'fruitfulness' is likely to mention unsuccessful proofs. He will do so if their method has borne fruit on other occasions.[1] The teacher may see other values in presenting error. Firstly, to show how one may learn from error, and secondly perhaps, to introduce the validity of the Hypothesisor's conception of mathematics being fallible. Throughout this chapter there have been clear indications that this perspective combines a way of teaching, with criteria for selecting syllabus content. On the one hand pupils are to be encouraged to look for[2] applications, and on the other hand material is chosen for the ease with which realistic examples can be drawn, provided that they link logically to the prerequisites determined by the overall syllabus objectives.[3] In these ways it is hoped that pupils will feel the reinforcement of personal achievement in area where applications are really fruitful.

CONCLUSION

At the beginning of this chapter, three problems were identified as likely to cause difficulties. The first was the concern mathematicians have that rigour ought not to be relaxed for the sake of applicability. The argument presented on pp.195-200 above, was that rigour is a pluralistic concept which only has sense with regard to a given logic, and so the

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1. An impression of this is given by Dieudonné in 'Abstraction in Mathematics and the Evolution of Algebra' in Learning and the Nature of Mathematics, pp. 100-13.

2. This reiterates the problem noted in the previous chapter on the genuineness of discovery. Ormell is not so concerned that the applications are not new, but that they are still hypothetical and have not already been adequately resolved. For example, fog is still a problem of motorway driving (Specimen Paper, 1974).

3. It would be unrealistic to deny that the most influential criterion of secondary syllabi remains, not moving too far from the most popular 'O' level and 'A' level syllabi at the time.

question of laxity may reduce to the more intractable one about the choice of logic, much to the surprise of mathematicians generally.

Secondly, hypothesisors like Ormell are denouncing as artificial the separation of Applied Mathematics from Pure Mathematics. They argue that there is no logical reason for such a separation if one denies the a priori status of mathematics, and that 'applicability' is equally meaningful internally as well as externally. This is the view of just one philosophy of mathematics and it has been shown in this chapter that if the concept of 'applicability' as well as 'rigour' is clarified then the limits of conflict and compatibility among the philosophical movements are clarified. Two moderate views of the technological perspective are thereby identified, rather than the extremes of 'all application' or 'all purity' that might otherwise be the basis of meaningless slogans.

Thirdly, concern can be shown for 'real' rather than 'pseudo' applicability even if one does not support the Strong Technology Perspective that hypothesisors hold to. Furthermore, there are still mathematics teachers who do identify a distinction of some kind between Pure and Applied Mathematics, and the argument of this chapter is of particular relevance to them. The argument has been that they should be at least concerned that their 'applied mathematics' involves 'rational applicability' as well as less logically compelling forms of applicative mathematics. In this way, the vital technological importance of mathematics as a modelling activity is open to all teachers to follow, and pupils can receive mathematical education rather than a training in particular routines.

Finally, the overall argument is that the technology perspective provides[1] the added dimension of 'fruitfulness', as compared with the other perspectives previously discussed.

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1. Naturally, the compatibility of this perspective with others will differ, depending on the philosophy of mathematics the individual teacher holds.

CHAPTER 11

MATHEMATICS TEACHING - INTER-DISCIPLINARY POSSIBILITIES

INTRODUCTION

In part 1 of the thesis four different views of mathematics were identified. So far, in this second part of the thesis, four different teaching perspectives have been discussed. It has been assumed that mathematics will be taught at any one time, clearly identified in terms of one of the movements presented in part 1 and according to one of the perspectives outlined in the previous four chapters. In this chapter, two alternatives to this view will be identified, and the first of these will provide the main focus of this chapter. These alternatives are:

1) Mathematics need not be taught as a separately identifiable subject, although particular skills traditionally linked to 'mathematics' will still be imparted. This will be called the 'Integrated Studies Possibility'.

2) Mathematics does not appear at all explicitly or implicitly in formal education. This is a logical possibility but as it will be argued in justifying mathematics education in part 3, not a realistic or desirable alternative.

A weaker form of the first alternative will also be considered:

3) Mathematics takes its place among other subjects to support a given theme or topic. Pupils are made aware of the distinct 'disciplines' and may be taught in groups labelled 'mathematics lesson' at some points in the week. This will be called the 'Inter-disciplinary Enquiry Possibility'.[1]

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1. In a project like the Schools Council Humanities Project, each unit calls upon an understanding of several areas, usually identified with separate content in the traditional curriculum, e.g. Drama and Geography in a Poverty Unit.

Before attempting a discussion of the strengths and weaknesses of 1) and 3), it is important to clarify what may be suggested by the phrases, 'integrated' and 'inter-disciplinary'. In his article, 'Curriculum Integration' (The Philosophy of Education, pp. 123-49), Pring identifies four uses of the term 'integration'. One of these four senses has wide-spread currency in education literature and White encapsulates this as the notion of the educated man's successful 'integration of what (he) learns into a total pattern of life' (Towards a Compulsory Curriculum,p. 82). In this sense, integration entails nothing about the curriculum beyond the ruling out of narrow specialism, and the requiring of the educated man's knowledge to be 'coherently organised' within his 'total pattern of life'. By 'integration' in this sense, one is simply supporting the generally accepted view that 'education is of the whole man'. This sense of 'integration' is neither particularly illuminating nor contentious, but the three other senses to which Pring refers, are more directly related to the curriculum, and epistemological questions in general.

In 1) above, the term 'integrated' could, according to Pring have two interpretations. On the one hand, it could mean that separating knowledge into disciplines is artificial, and a unified presentation is therefore preferable; taking this argument to its extreme, it is that knowledge is a 'seamless whole' and so any development at any one point, will necessarily influence knowledge at other points in the unified body. Pring calls this 'the "strong" thesis' (p. 128) and contrasts it with 'the "weak" thesis', introduced on the same page, that limits the 'seamless whole' to some sub-section of 'knowledge', like 'humanities' or 'the sciences' or 'the arts'. Clearly the phrase 'Integrated Studies' found in 1) above, could be taken in either sense.

In 3) above, the phrase, 'Inter-disciplinary Enquiry' replaces 'Integrated Studies', and Pring also makes a similar conceptual distinction. Pring sees 'Inter-disciplinary Enquiry' as part of 'a claim for a closer examination of the logical interdependence of different disciplines while at the same time recognizing their distinctiveness' (p. 128). Pring concludes on p. 148 that this 'interdependence' could be highlighted in the curriculum as a whole, without anyone pursuing Inter-disciplinary Enquiry as a teaching method. In other words, the phrase 'Inter-disciplinary Enquiry' may indicate an approach to education in general, or a focusing in on a particular topic, problem or theme, by several disciplines. The latter is most commonly taken to be the novel idea of Inter-disciplinary Enquiry.[1]

Thus four senses of 'integration' have been identified, and the last three will be referred to in the remainder of this chapter. The four senses may be characterised as:

1) The 'education of the whole man' thesis,
2) The 'strong integration' thesis,
3) The 'weak integration' thesis, and
4) The 'inter-disciplinary enquiry' thesis.

The fourth position has been shown to be itself ambiguous. Pring seems to identify three senses of 'inter-disciplinary enquiry':

a) in which logical inter-dependence of different disciplines is accepted, but their distinctiveness is also recognised. Both points influence an overall education policy of Inter-disciplinary Enquiry.

b) in which a given topic, problem or theme is presented in a lesson or series of lessons by a teacher or teachers who are concerned to interrelate otherwise separated disciplines.

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1. Logic may require both the policy and the implementation through the particular novel teaching approach, but this would still not guarantee that teachers were aware of both policy and teaching method.

c) in which all inter-dependence of disciplines is seen as contingent,[1] and any policy of Inter-disciplinary Enquiry is supported only if it has 'success'; for example, in terms of improved examination results or employment prospects.

While a) and c) are, or are parts of, education policies, b) is a teaching method, and as they stand, a), b) and c) are not necessarily interconnected. a) and c) require teachers to show links among disciplines, but only b) requires the teacher or teachers to be actively involved in the imparting of knowledge from differing disciplines in their own lessons.

In this chapter, discussion will begin with the outlining of the arguments in favour of the discreteness of mathematics and how these arguments reflect upon views for and against the retaining of mathematics' position as formally discrete within most usual notions of education. This will lead to further analyses of the 'integration theses' just outlined, and finally the compatibility of these theses with the teaching perspectives discussed in the earlier chapters of this part of this thesis, will be considered. This will leave certain points about mathematics teachers in general, to be brought together in the Conclusion to this part of the thesis, and certain points about the training of mathematics educators to be resolved in the final part of the thesis.

THE FORMAL DISCRETENESS OF MATHEMATICS

In one sense the formal arguments were presented in part 1 of the thesis and so do not require repetition. What may be useful is to bring together the features of those arguments that seem most pertinent to the school situation. A contemporary model for such a pertinent argument would be the criteria to which Hirst has referred in his works. Drawing

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1. Pring seems to be unsure if Hirst would hold position c), but some clarification may be found in Hirst's recent letter to the Journal of Further and Higher Education, 4 (1), Spring 1980, pp. 122-3, in which he states 'forms have so many complex inter-relations', 'differences within any one form', but between forms one is making 'a categorial shift'. This does seem to leave c) alone as a possibility.

on the most famous of his articles and also one of his latest,[1] three criteria push through to the front. In identifying a form of knowledge one must isolate 1) A Test for Truth; 2) Uniqueness of Meaning - 'a kind of knowledge which, of its nature, cannot be expressed in any other symbolic system' (Article B, p. 107); 3) Logical Structure by which the form of knowledge develops. This model fits well the likely response from Logicists.

For the Logicist, 'Logic and Mathematics' form an isolatable area of formal derivability from self-evident axioms or formal abstractions. Thus Logic and Mathematics clearly consist of a distinctive logical structure which can only develop by further deductions from what has been previously proved. The deducibility within the system that structures the development of the form of knowledge also provides the criteria for truth. At each stopping point what is derived from self-evident axioms is a valid theorem. That it is also true is known by its necessary correspondence to some part of an absolute reality (Frege's 'realistic logicism') or to some nominal reality of abstractions (Russell's 'nominalistic logicism')[2] which provide 'meaning' through reference, for the elements of mathematics. Thus, not surprisingly, the Logicist has a view of logic and mathematics in which the three Hirstian criteria are identifiable but logically interconnected. 1) Truth rests on deducibility from axioms; 2) There is a Referential Theory of Meaning where the reference is to an unique body of concepts; 3) The test for truth is simultaneously the means by which the form of knowledge develops. These conditions are sufficient to make mathematics discrete, for any new system meeting all three would be a part of logic and mathematics.

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1. Article A: 'Liberal Education and the Nature of Knowledge' in Philosophical Analysis and Education, pp. 113-38.
   Article B: 'Human Movement, Knowledge and Education' in Journ. of Phil. of Ed., Vol. 13, pp. 101-8.
2. The distinction was more fully discussed above, part 1, Chapter 2.

All Formalists[1] argue that mathematics is separable from logic because the latter only is based upon self-evident axioms. While mathematics develops according to rules rather as logic does, it does so from posited foundations and the objects of manipulation are not logical but empirical. To the Formalist a mathematical proposition consists of a series of strokes and has no further reference. To the Hilbertian formalists the criteria for the discreteness of mathematics would be coincident with those of the Bourbaki, except that the latter make no claims either way on the a priori/a posteriori nature of mathematical knowledge. To the Strict Formalists, one of Hirst's criteria is considered inapplicable for mathematics. Mathematics is 'a kind of knowledge which, of its nature, cannot be expressed in any other symbolic system' but there is no conception of 'meaningfulness' in this system. Its discreteness is identifiable in its having neither meaning nor self-evidence. The presence of the former would make it an empirical science, and of the latter, logic.

Now in part 1 of the thesis a movement was identified that neither tied mathematics to logic, nor reduced the distinction between mathematics and other sciences to the incorrigibility of the former's calculations. This movement was Intuitionism but there are others who would argue for

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1. Formalisers, like the Bourbaki, also support the view that mathematics is separable from logic. They see the axioms of mathematics as human constructs that are derived from the set theoretical framework which will be assumed internally consistent until proved otherwise. Logic alone is considered as consisting of self-evident axioms, and as such is separable from systems based on set theory. By formal derivability, various systems are produced from set theory and according to how one uses a given system it receives a name and 'meaning'. A system entering mathematics as 'algebra' might go into some secret service as an encoding/decoding system. Thus mathematics is viewed by the Formalisers as set theory which has been given a particular use, and that is what is meant by calling it an 'interpreted' or 'meaningful' system. Mathematics is a form of knowledge for its test for truth and method of development are both based upon 'formal derivability' from set theory and its distinctive use is its peculiar 'meaning'. Bourbaki was introduced on p. 53, footnote 1 above, and some of these points were made then.

the discreteness of mathematics from logic and the natural sciences, in its being both '<u>a priori</u>' and 'synthetic'. Kant, two hundred years ago, and Koerner today, have held this position but would not call themselves Intuitionists. Kant held that mathematics provides the categorial skeleton into which concepts of communication are fitted (the 'stuffing' of the world - the objects, as against the stage of space and time). Koerner holds a different argument for the discreteness of mathematics. He specifically goes out to show how it differs on the one hand from logic and on the other, from the natural sciences. Firstly, the propositions of logic are unique particulars, while those of mathematics are generalisations. Thus in logic one could not consistently assert both 'p implies q' and 'p implies not-q'. These propositions are incompatible in the sense that both could not be part of 'logic'. Koerner argues that the mathematician can frame incompatible sequences of words without committing a logical contradiction. He can claim that both 'there exists an object such that it is the square root of -1' and 'there does not exist an object such that it is the square root of -1' hold in mathematics for they are rules[1] that tell one how to proceed in the game and not assertions. It is comparable to the fact that both the rules 'Eat fish on Friday' and 'Don't eat fish on Friday' may be present in the same language-game without any contradiction resulting. The contradiction arises only if one attempts to obey both rules simultaneously. Koerner is stressing that existential propositions in mathematics are logically different from those in the natural sciences, for mathematical objects are human constructs. The objects of mathematics exist or cease to exist according to the rules of the game. In this sense, they are logically different from empirical objects. Furthermore, Koerner argues that in mathematics there logically

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1. Wittgenstein took a similar position in <u>Lectures on the Foundations of Mathematics</u>, pp. 248-52.

cannot be the problem of ambiguities that occurs with empirical objects. Empirical objects can be such that it is disputable whether it is 'bluey-green' or 'greeny-blue' but it cannot logically be disputable whether 2347953 is a prime number or not. 'Inexactness' has no place in mathematics.

The Intuitionists have their own arguments for the discrete position of mathematics. Koerner has not provided his own explanation of the formal nature of mathematics but is only independent of the Formalists in his conception of the foundations. They are not self-evident or abstractions but 'rules', 'rule-incorporated objects'. Mathematics still stands out as a system of formal derivability as it does for the Formalists. The Intuitionist on the other hand provides distinct criteria for truth, meaning and mathematical development.

1) Truth rests on 'constructibility' and not formal derivability.

2) The study of mathematics may be seen as describing human thought processes. The objects of mathematics for the Intuitionists are mental and cannot logically be seen as empirical objects precisely because in any such form, 'we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts' (Heyting, op. cit., p. 4). The Intuitionists argue that there is a subject, mathematics which constructs its own framework, an evolving system, and its constructions are mental rather than empirical. Such a subject cannot be identified with any other because of its evolving nature and yet its demand upon the mind may give it a potential use in describing thought processes or more likely, encouraging thought processes. In other words, mathematics may be justified educationally because it is in this area that 'thinking'[1] is a precondition of 'knowledge'. If 'mathematics' is logically identified with 'thinking' then in terms of problem 2, p. 181 and comment on p. 208,

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1. Clarification is presented as logically impossible by the Intuitionist argument that the descriptions in Intuitionistic logic(s) are no better than a model of actual thinking. While studying mathematics, brings an appreciation of thinking, thinking is only one group of functions of the human mind.

'if children <u>ought</u> to be made to think, then studying mathematics to acquire knowledge of mathematics will necessarily involve children thinking'.

3) The logical framework develops as the subject develops and so this structure cannot logically be identified as the structure of any other subject. Not even mathematicians know for certain what the logic of mathematics is, but only what is a suitable logic to describe what has already been constructed. As Wittgenstein concisely but disparagingly put it, 'Intuitionism comes to saying that you can make a new rule at each point. It requires that we have an intuition at each step in calculation, at each application of a rule; ...We might as well say that we need, not an intuition at each step, but a decision.' (<u>op. cit</u>., p. 237). Educationally, one must get pupils 'inside mathematics' for them to understand it. There is no substitute for 'doing'. One cannot pick-up mathematical knowledge second-hand.

The <u>Hypothesisors</u> retain the view that mathematics is empirical but formalisable.[1] However 'formal derivability' is not a sufficient definition of its test for truth. It must also demonstrate 'fruitful applicability' which simultaneously identifies the source of its own development when the applicability is 'internal'. The other distinctive feature comes through in the meaningfulness of mathematics. Mathematics is not meaningful in the sense that other sciences are normally thought to be; for it does not describe some reality but 'previews' it. It enables 'us to understand the possibilities of the real world' and not just to describe what has already happened or could easily be constructed (See Ormell's 'Towards a naturalistic mathematics in the sixth form' in <u>Physics Education</u>, July 1975). Mathematics, claims Ormell, is the product of

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1. The nature of this formalisability was identified in part 1 Chapter 4 where it was made clear that Lakatos may have held a fundamentally more relativist position than Ormell does. Ormell holds unflinchingly to 'bivalence'.

'disciplined imagination', and of 'thought-experiments' in Lakatos' terms: it is not simply the manipulation of empirical marks on paper as the strict Formalist would suggest. Both Hilbertian and Curry-type Formalist views of mathematics are rejected by the Hypothesisors who identify mathematics as always open to interpretation, always the potential model.

From each philosophical movement comes an organised justification of the discreteness of mathematics, and any one set of criteria can be employed to identify a subject to be placed on the time-table. One could go on then and find sociological, psychological and historical reasons that would give added strength to mathematics' claim for an independent place on the time-table. This is not a part of this thesis however. Yet one set of discreteness criteria, not strictly philosophical will be discussed, for in them seem to lie the root of a justification of mathematics' employment in Inter-disciplinary Enquiry, in a sense stronger than sense c), described on p. 212 above. In the work of Wilder one finds an explanation of the appropriateness of mathematics not being seen as some independent fortress.

'Mathematics is something that man himself creates and the type of mathematics he works out is just as much a function of the cultural demands of the time as any of his other adaptive mechanisms.' (Wilder, <u>Evolution of Mathematical Concepts</u>, p. 3). This is not just a picture of mathematics as a set of tools that change according to evolutionary requirements like any applied science, for mathematics is special, it is also the source of extensive aesthetic experiences. It has a peculiar duality of 'science' and 'art, 'inventor' and 'creator'. Here lies the discreteness of mathematics. Wilder who takes an anthropological approach, does seem to rest his arguments on the greater importance of mathematics as applicative either 'in all physical theory' or 'within mathematics'. 'Even the purest of mathematics may suddenly find application' (<u>op. cit</u>., p. 186). Wilder's criteria are not purely contingent for ten pages later, he says the 'major difference between mathematics and the other sciences, natural and social, is that whereas the latter are directly restricted in their purview by environmental phenomena of a physical or social nature, mathematics is subject only indirectly to such limitations'. However on the following page, he brings one back again from the edge of a logically necessary criterion when he goes on, 'Yet perfect rigour and absolute freedom from contradiction in mathematics are no more to be expected than are final and exact explanations of natural and social phenomena' by other sciences. All 'continue to evolve more abstract, scientifically effective, and marvellous concepts'. Wilder is thus going on to stress what all developing areas of scientific study share, 'abstraction'.[1] Wilder turns this notion which would commonly be used as a criterion of discreteness for mathematics into a criterion of possibilities of integration. All knowledge may be seen as developing towards this 'abstract ideal state'. In evolution, every theory becomes a 'model'[2] and mathematics is seen as the science that initiates models which come to find applications in theories and are then preserved as models in the area to which they were applied, once the theories in which they became embodied become modified. Wilder seems to identify an evolutionary cycle of 'Mathematical Model - Scientific Theory - Scientific Model'.

Wilder takes the view that evolutionary forces drive men of science across boundaries. Wilder is free from the fetters of any philosophical movement, and perhaps even of logical consistency, but he does provide

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1. With Lakatos in mind 'abstraction' seems to be the social scientist's term for what philosophers call 'formalisability'. The key to relativism for both Wilder and Lakatos lies here. Ormell uses 'abstraction' to identify mathematics' power to explain science. ('Mathematical Models and Understanding in Science', an address to the A.S.E. Conference, Reading, January 1979).

2. 'Theory' and 'model' are used in the sense noted on p. 159 above, where the former is thought to be a true description and the latter is known not to be.

an interesting argument for a realistic appraisal of inter-disciplinary enquiry involving mathematics. Even though the logical discreteness arguments become blurred in Wilder's hands, he has still presented mathematics as 'distinctive'; bridging 'arts' and 'sciences', for example. This exemplifies well the sense a) upon which Pring focuses ('Curriculum Integration', p. 148, and above p. 211), when he talks of a concern for 'interrelations'. Given that all the positions discussed here, including Wilder's, support some notion of mathematical discreteness, it would seem to be appropriate now to try to identify in what sense one can still talk of, 'inter-disciplinary mathematics teaching'.

INTER-DISCIPLINARY MATHEMATICS TEACHING

Through the 1960s and 1970s two forms of Inter-disciplinary mathematics teaching have developed. On the one hand, there is the servicing of other areas by mathematics. This is well exemplified in the increasing weight of statistics in geography syllabuses. On the other hand, there is the inclusion of real life situations in mathematics syllabuses, as occurs when football league tables, fashion and food budgeting are included.[1] In each case, the pupil may be encouraged to bring together knowledge from other compartments of his mind than that which he usually calls upon when entering the given time-tabled lesson.

The State of Inter-disciplinary Mathematics Teaching. Although there have been these developments it is very difficult in practice to find in most secondary schools anything more adventurous than examples like those just given.

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1. These areas may facilitate pupils' success in taking C.S.E. in Physical Education, or any Home Economics' public examination with a consumer affairs element. In Aspects of Secondary Education in England, p. 150, one reads, 'some girls were learning, as part of their mathematics, to enlarge skirt and dress patterns...and they were later to make them in the needlework department'.

Mathematics may appear however in an Inter-disciplinary teaching context without its being taught there. Mathematics may be taught in mathematics lessons and used in physics lessons. The science teacher mentioned on p. 141 above, teaches those parts of mathematics which he requires for his physics, like density, and not because he has chosen the road to Inter-disciplinary Enquiry. Thus mathematics may appear out of context for three different reasons:

1) a subject teacher finds the pupils ignorant of a mathematical tool which he requires and so the subject teacher provides the tool.

2) the pupils' knowledge of certain mathematical tools is required to develop a particular Inter-disciplinary theme and so mathematics is used there.

3) mathematics features in the objectives of the planning of the Inter-disciplinary theme and the pupils are taught some mathematics in that light.

In 2) the teacher intends to identify new uses of these mathematical tools but does not intend to introduce new mathematical tools. This approach is particularly compatible with the 'idealisation' view of applicability picked out in the previous chapter. Mathematical tools meet requirements set by some problem from either another subject or group of subjects and in solving the problem it is not mathematics itself which develops but this other subject or group of subjects. This may be contrasted with the 'inter-action'[1] approach which claims that the use of mathematics must involve some modifications to both the tools and to what they are applied. The implication for education is that in the former case there is learning about mathematics only, while in the latter case, there is

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1. See p. 193 above, where 'idealisation' and 'inter-action' were introduced.

the possibility of learning in mathematics also.[1]

This contrast is still essentially centred in a discussion of teaching perspectives rather than necessarily involving different philosophies of mathematics. However the 'idealisation'/'inter-action' distinction does signpost certain philosophical conditions for Inter-disciplinary Enquiry and Integration. If these thematic approaches involve the sweeping away of 'knowledge boundaries' then certain stumbling blocks will arise. The most obvious block is that between '*a priori*' and '*a posteriori*' knowledge. If mathematics is taken to belong to the former category then necessarily any 'problem-solving' can only go in one direction - no *a posteriori* tools can be used in mathematics. Mathematics is seen to 'idealise' reality but not to 'inter-act' with it. The approaches identified in this chapter can only initiate knowledge and understanding of mathematics in any full sense if an 'inter-active' approach is accepted. Only the logicists must of necessity reject this choice, for even the intuitionists can choose at the level of Intuitionistic logic(s). At the other extreme to the Logicists would be those committed to a Pragmatic Theory of Truth who would argue that the essential position of 'inter-action' in the concept of truth,[2] leads naturally, if not of logical necessity, to Inter-disciplinary Enquiry.

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1. Given the distinction Pring draws between 'integration' on the weak thesis and 'interrelationships among disciplines' then the distinction here between 'inter-action' and 'idealisation' can be seen to parallel it. Pring argues that '"integration" raises certain questions in epistemology to which "interdisciplinary" remains indifferent' ('Curriculum Integration', p. 135, *loc. cit*.). Similarly 'inter-action' makes epistemological demands that only certain philosophical movements can logically accept, while 'idealisation' makes no such demands. 'Inter-action' goes some way towards Pring's notion of 'integration' but stops short of being for the Hypothesisors at least, a feature of 'supra-subjects' as Pring sees 'integration' identifying. Pring does have another notion of 'integration' which has strong similarities with the views of the Hypothesisors. The link is not surprising for Pring's 'Problem-solving Method' is tied to Dewey, and the Hypothesisors have strong ties with Pragmatism generally (see 'Curriculum Integration', pp. 143-6, and I.D.E., sense a) given on p. 211 above).
2. Pring makes precisely this point, that it 'imports a notion of truth quite foreign to our normal language', i.e. one form of such Enquiry is most naturally rooted in Pragmatism.

The limiting constraint is whether or not a mathematical proposition can satisfactorily demonstrate its warranted assertibility by applications <u>within</u> mathematics or must it prove itself in the <u>outside</u> reality. Whatever one's position on this point, only the Hypothesisors of all the philosophical movements discussed in part 1 of the thesis would seem to give support with some logical, rather than evaluative force, to Inter-disciplinary Enquiry, in this sense.

<u>The Responsibility of a Teacher for His Subject</u>. If the teaching situation involves inter-disciplinary enquiry then other teachers than specialists in a particular subject will be concerned about that subject. In the example on p. 220 above, the geography teacher is concerned about statistics and the teaching of statistics. In a genuinely 'inter-active' situation, it would seem reasonable for the mathematics teacher to care too.

In his article, 'Teachers and the Curriculum', Wain does point in this direction but concludes with the less revolutionary hope that 'teachers will increasingly take on much more responsibility for the development of their own subject and other related subjects in schools' (<u>Mathematical Education</u>, p. 155). In 1979 terms, one ponders the problem that mathematics is often taught by those teachers of 'other related subjects', for there is no one else to do so, and one concludes that it is the responsibility of those mathematics educators that there are, to be responsible for how their subject is taught. Wain concisely describes what has happened but does not grasp this focal nettle.

> The implied assumption that mathematics should be a compulsory subject with a large allocation of time should be looked at closely...The reform of the mathematics syllabus has gone ahead in an isolated way...there has been an increased tendency for mathematics in particular to be shut into its own water-tight compartment...mathematics seems to have raised even higher the wall that traditionally has separated it from other parts of the curriculum. This has happened at a time when the applications of the subject have widened remarkably into areas formerly thought of as non-mathematical. It is often also the case that, in schools where integration of subjects and moves to mixed ability teaching have occurred, the mathematics

> teachers often plead a special case for their subject as not being capable of fitting in to such new structures. This is not to say that such developments are to be applauded but rather to indicate that mathematics is usually one of the subjects least able to accommodate itself. (ibid., p. 152).

Thus 'inter-disciplinary mathematics teaching' can mean more than mathematics having a part to play in a topic with other disciplines, and more than the mathematics teacher making mention of other subjects in his lessons: it could also mean the mathematics teacher feeling a responsibility for any kind of mathematics educating going on in the institution in which he works, at least.

ARGUMENTS FOR PRESERVING 'MATHEMATICS' AS A SEPARATELY IDENTIFIED SUBJECT.

The most simplistic, if not the weakest of arguments is that headmasters continue to put such a label on the time-table. However, headmasters themselves have made a selection and one must focus on the reason for their choice. They may say,

1) Mathematics traditionally appears on school time-tables, give me a reason for change.
2) All interested parties like employers, parents and governors expect it.
3) Pupils who leave school, apply to bodies which often expect or require the successful prior study of mathematics. The label makes it clear to both pupils and these other bodies that the school is explicitly concerned to meet the requirement.
4) Pupils learn best at lessons with 'expected' labels, in the long term. Novelty is not obviously compatible with education. Furthermore, the secondary label, 'mathematics' has probably been presented in the Junior school already.

5) Schools have a duty to ensure knowledge is imparted and one area of knowledge is[1] mathematics,

AND a) this organisation of knowledge is shown most simply by direct labelling;

OR b) the areas of knowledge cannot <u>logically</u> be imparted except in isolation, so what sense could be made of alternative or no labelling.

These are just the kinds of reasons that one may collect from headmasters but given that in Great Britain there is no legal compulsion to have mathematics as a label and nothing about content follows from the concept of a time-table, there cannot be a <u>logically</u> compelling argument for the label's appearance in schools.

Such arguments are not of central importance to this thesis. What is of importance is that such arguments lead to a retention of the label 'mathematics' on the time-table but those who present the arguments may have radically differing conceptions of mathematics from one another and from the philosophers of part 1 of this thesis. Much of the discussion that has occurred in this part of the thesis has led to the conclusion that what is defined by philosophers and professional mathematicians as

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1. A critical point hangs on whether the 'is' is logical or contingent. Clearly to Hirst it is logical while to anti-realists like Dewey the link is contingent, depending on its practical success (see part 1 Chapter 4 and Chapter 5, pp.85-90). A more extreme relativism is to be found in the writings of M. F. D. Young. This epistemological ambivalence has been central to the earlier discussions of the Hypothesisors and well-characterised in the work of Lakatos. He presents a picture of mathematics separated necessarily from other areas of knowledge by logical conditions like the employment of 'thought-experiments'. This is not a sufficient justification and Lakatos provides this through the empirical assertion that mathematics is more thoroughly formalisable than other sciences. More extreme relativism is not discussed in this thesis because there is only minimal documentation of such a perspective influencing mathematics teaching. As a matter of fact, mathematics remains strongly independent, only open to attack from the 'inter-action' of applicability discussed in the previous chapter. That it has no more to fear at this moment is highlighted by the complete absence of the language of 'integration' in such documents as the Schools Council report, <u>Mixed-ability teaching in mathematics</u>, 1977.

mathematics cannot be achieved by most pupils in schools, before the compulsory schooling has ended.[1] The formal demands and the pedagogic reality conflict. It is probably the case that many of those seeking the permanent position of mathematics on the time-table have a notion of mathematics that would not be found in part 1 of this thesis, at all. There are still those who equate mathematics with 'arithmetic and correct calculations' but this is no longer the case with the industrially and commercially linked public examinations of City and Guilds and the Mathematics in Education and Industry Project. On p. 78 of <u>Mathematics through School</u> Geoffrey Matthews draws together many of the attitudes inherent in what one might call 'progressive mathematics teaching'. Focusing in on the concern 'to relate the mathematics with real life', he says,

> School mathematics must keep one eye on the outside world... start looking across the boundaries...But in our enthusiasm we must not submerge mathematics totally within the general curriculum. There are still topics which are straight <u>mathematics</u> and there are still times when honest practice is necessary...if,...it grows naturally from the environment the children will also grow to enjoy the subject and to thrive at it.

Matthews identifies three parts to mathematics:

a) That it is able to look across boundaries - i.e. it has applicability.

b) That there are self-contained elements of 'straight mathematics'.

c) That certain parts of mathematics are such that familiarity is most readily achieved through 'honest practice'.

When different people discuss the essential place of mathematics on the school time-table there is probably agreement on the fundamental importance of at least one of these parts. The headmaster may stress b) and c) while the employer/governor may wish to stress a) and c), and so

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1. See the discussion of 'creative work' in mathematics on pp. 127-31 above and of the formal demands of 'the game of set theory' on pp. 150-54 above.

on. However, many of the philosophers and professional mathematicians with whose views much of part 1 of the thesis was concerned, may not go beyond recognising the relevance of b). They identify mathematics with a level of conceptual development rarely attained before thirteen, 'a system of formal operations'. Thus the mathematics of the early years of the secondary school which pupils may appreciate because it stays close to a) and c), is not recognisable as mathematics to these philosophers and professional mathematicians.

These people would argue that the word 'mathematics' isolates a concept whose technical usage logically excludes reference to an amalgam. If mathematics occurs in an inter-disciplinary curriculum then its characteristic features must still be identifiable. Each philosophical movement discussed in part 1 of this thesis, asserts the discreteness of mathematics, according to logically compelling criteria.[1] In 1969 Matthews wrote that mathematics teaching ought 'to help the children develop gradually - and not overnight - from discovery with things to eventual abstraction with pencil and paper' ('Mathematics in the Middle Years' in <u>Schools Council Working Paper No. 22</u>, p. 64). The important point is that inter-disciplinary enquiry is not ruled out, but the teacher must have a full-blooded appreciation of mathematics as a discrete discipline, in order to produce materials that still incorporate the critical features of mathematics. The label 'mathematics' on a time-table may be no more than a signpost for most pupils, in that they do not possess knowledge and understanding of what delineates mathematics, but this is not an acceptable state for all teachers using mathematics to be in. Pupils can feel that they are following a trail that will lead

1. In Chapters 2 to 4 of part 1 of the thesis, each philosophical movement gave a response to the question, 'Is "mathematics" logically distinct from the "empirical sciences"?' and in so doing particular criteria were identified as characterising mathematics for that movement.

eventually to an understanding of mathematics, if taken for long enough, but for those who associate mathematics with 'a system of formal operations' it is logically unacceptable for teachers of mathematics still to be on the trail.

Even mathematics educators like Wilder, who have a 'cultural view' of mathematics identify the subject as at least distinguishable by its independence from direct empirical influences. There may have been a move away, in the last ten years, from 'mathematics taught as intrinsically valuable' to 'mathematics taught for its usefulness', but there has been no sign of a move from 'mathematics taught only for intrinsic value' to 'mathematics taught only for instrumental reasons'. Furthermore, the hypothesisors may ease the criteria by which a pupil can be said to have a real understanding of mathematics, but they are at one with the other philosophical movements in requiring anyone who imparts mathematics, like a teacher, to be able to demonstrate knowledge and understanding of the subject.[1]

All the views that arise from the philosophical movements of part 1 of this thesis agree that <u>integration</u> of mathematics in either the strong or weak senses given on pp. 210-11 above are logically impossible, with the exception of the logicist accepting a weak form of integration in the unifying of mathematics with logic. Although the strict formalists talk of mathematics as 'uninterpreted science' and hypothesisors like Ormell call mathematics 'the science of possibility', they refrain from claiming that mathematics has been reduced to some other discipline. The criteria of discreteness are seen to hold mathematics solidly apart from other science areas.[2] This is not a rejection of the possibility of

1. This point is taken up in the general discussion of what is required in educating mathematics teachers, part 3 below.

2. Put explicitly 'integrated studies including mathematics' is seen as self-contradictory for there cannot be a mathematical method apart from mathematics itself, as there can be 'scientific method' apart from any particular science. This was shown in part 2, Chapter 9.

<u>Inter-disciplinary Enquiry</u>, in any of the senses outlined on p. 211 above, but the technology perspective of the previous chapter does seem to stand a shorter step from the policy described in sense a)[1] on p. 211, than any of the other orientations discussed in this part of the thesis. However no perspective rules out this kind of approach.

<u>CONCLUSION</u>

Two fundamental conclusions would seem to be derivable from the discussion in this chapter. The first refers to integration, and the second to inter-disciplinary enquiry.

1) There would seem to be a unified rejection of integration if it is taken to imply the presentation of knowledge as a 'seamless whole'. White summarises the argument most precisely when he writes that there 'may or may not be a case for teaching the different sciences - physics, chemistry, biology - under the one rubric of 'general science': if 'integration' only goes as far as this, it is at least binding together disciplines which have important structural features in common. But in so far as it goes beyond this to pulp everything together, it is unintelligible.' (White, <u>op. cit</u>., p. 89). In the last thirty-five years arithmetic, geometry and algebra have come together under one secondary school examination syllabus, but to talk of the integration of mathematics with other areas of knowledge would seem to be the call of a lone voice in the wilderness.

2) Three senses of 'inter-disciplinary' were introduced on p. 211. <u>Sense c</u>) presented no obstacles to any philosophical movement or teaching

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1. In sense a) there is 'logical interdependence of different disciplines' as a feature of an Inter-disciplinary Enquiry education policy, and if 'interdependence' is interpreted as 'interaction' was in the previous chapter then there would seem to be close ties with the technology perspective. Furthermore, such an interpretation would restrict the acceptability of such a policy to other philosophical movements, particularly the logicists for whom 'interaction' is limited to other areas of <u>a priori</u> knowledge only. Naturally if 'interdependence' is taken in some weaker sense, as that suggested by 'idealisation' in the previous chapter, this constraint would not arise.

perspective for it simply suggests that if the reference to other disciplines while teaching a given area, perhaps mathematics, leads to 'better results' (according to one's own particular measure), then this would seem to be reason for considering such an approach. Sense b) which requires teaching a given topic, problem or theme by interrelating disciplines, would seem also to be acceptable to any philosophical movement or teaching perspective. Furthermore, the discussions in the previous chapters might themselves provide teachers with stimuli for Inter-disciplinary themes; like 'Mathematics and the Arts'. In the story of Chapter 6 of this part of the thesis the young teacher used an 'Art perspective' in the sixth form (see p.100 above) and a 'Games perspective' with first formers (see p. 101), and so on. The range is freely available to the teacher, and philosophy alone cannot either make, or stop the teacher making, certain choices. Clearly, if a teacher holds that all or any branches of mathematics ought only to be taught through a single given perspective, then the range of Inter-disciplinary contexts will be thereby limited. The theme of 'war' ought not to be presented from a games perspective. The perspectives of the previous four chapters cannot be more restrictive than this. Sense c) is the only form of Inter-disciplinary Enquiry that does seem to entail constraints on its acceptability to differing philosophical movements. Although it does not involve the reduction of one or more areas of knowledge to some kind of 'seamless whole', it does seem to challenge the isolationism of philosophies that see mathematics as either logically untouchable by empirical influences, like Logicism, or those that see mathematics as uninterpreted, like Formalism.

'Integration' may be rejected as an education policy and teaching method for mathematics, but within given constraints 'Inter-disciplinary Enquiry' has been seen to provide a valid addition to the teaching perspectives discussed in the previous chapters.

CONCLUSION TO PART 2

INTRODUCTION

This part of the thesis has taken mathematics into the school and considered the ways in which the subject might be presented. Four perspectives have been outlined: 'mathematics as an art-form', 'mathematics as a game', 'mathematics as a science' and 'mathematics as a technology'. The first perspective has been seen to differ according to one's conception of aesthetic criticism. Three such conceptions have been given and these were found to have similarities with differing philosophical movements discussed in part 1 of this thesis. While the discussion of the other perspectives has not shown the same breadth of interpretation, differing philosophies of mathematics from part 1 of the thesis have shown themselves more readily aligned to one perspective rather than another. Although a teacher's view of the nature of mathematics may lead him to teach mathematics from one perspective rather than another, it has been mentioned on several occasions in this part of the thesis, that a teacher may have no such philosophical thoughts in view, but be influenced by psychological findings or just by his instinctive belief that one approach has been found by him to be more successful than another. One important factor in the third part of this thesis is a consideration of the dangers and limitations of such 'instinctive' decision-making.

Part 2 of the thesis has been introduced through the use of an imaginary young mathematics teacher who attempts to prepare himself for the coming year in which he will teach a range of ages and abilities. Through this teacher the four perspectives are introduced in a secondary school setting. In the main body of this second part of the thesis, each perspective is taken separately and considered in the light of three questions:

1) What distinguishes one perspective from all the others?
2) Must one logically have come to appreciate one perspective as a pupil, before one can appreciate some other perspective?
3) Does one find a common logical sequence to what is taught, no matter through which approach it is taught?

Finally the possibility of teaching mathematics through a policy of inter-disciplinary enquiry was considered and favourably noted, provided one has made very clear what is taken as the meaning of 'inter-disciplinary' and that certainly it should not imply the taking of knowledge as a 'seamless whole'.

THREE QUESTIONS

The first question was successfully answered in the sense that each perspective is differentiated without one perspective collapsing irretrievably into any other. However, a particularly close connection was found between the science perspective and the technology perspective, for 'technology' is normally taken to mean 'applied science', or at least, 'applying a scientific method', and as such the technology perspective is an extension of the science perspective. This leads naturally to the second question as to whether or not a pupil can appreciate the one perspective apart from the other. The technology perspective does logically require appreciation of the science perspective, but this need have no temporal implications. In gaining an awareness of the technology perspective a pupil could simultaneously gain an understanding of a scientific method. No such close relationship was found between any other pair of perspectives, but conversely it was found that if one considered mathematics is a game then one could not logically be also considering that mathematics is a science. The former perspective rejects the notion of 'seriousness' while that of the science and technology perspectives requires it. Naturally, if one is only presenting mathematics as metaphorically a game or even slightly more tightly, if one is arguing that mathematics might be viewed as a game, then no such logical problem arises. Similarly, teachers following the technology perspective were seen to be at odds with any perspective that showed an overriding concern for the aesthetic qualities of mathematics, at the expense of questions of internal or external usefulness.

The attempts to answer the third question have made it clear that any more than generalised responses are impossible until the aims and objectives of a teacher's or a department's or a school's mathematics education are known. Once these aims and more specifically the objectives are known, then one could say that this objective can only be achieved if that objective is also achieved. Only on this basis could one go on to give any logical sequence to what is taught. It is not the intention of this thesis to do that, but in part 3 of the thesis, the focus is put on the notion of 'mathematics education', and its aims. This leads to consideration of the implications for the education of mathematics educators, and their attempts in future years consistently to produce objectives for mathematics education and a logic to what will be taught. Clearly, part 3 can only be accomplished if parts 1 and 2 of this thesis have successfully clarified the range of interpretations found in both the philosophy of mathematics and of mathematics teaching.

CONCLUSION

In this final section of part 2 of this thesis, opportunity has been taken to look back beyond this part of the thesis and on to the final part of the thesis, but it would seem worth identifying finally the two critical pointers for the planning of mathematics education that have come to the fore in this part.

Firstly, it has been shown that there are at least four realistic perspectives through which mathematics may be taught. These may be used to highlight differing features of mathematics, provided that the pupil is not given the view that mathematics is and could only be a game, or a science, say.[1]

Secondly, it has been shown that the philosophical movements introduced in part 1 of the thesis are compatible with mathematics being taught from a variety of perspectives, and even compatible with mathematics taught within an inter-disciplinary approach.

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1. The argument of this thesis is that any teacher who did present mathematics as exclusively a game would be cutting off choices from his pupils that would require a level of justification that this author has not found warranted. This point is considered in more detail in the following part of the thesis.

PART 3: MATHEMATICS EDUCATION,
ITS AIMS AND RESPONSIBILITIES

INTRODUCTION

In this final part of the thesis, two contrasting views of the nature of aims in education are outlined. This same contrast is then identified in mathematics education. The distinction is drawn between fundamental aims founded on intrinsic values as against aims founded on extrinsic values. The clarification of this distinction will be the first objective of this part of the thesis.

Once the distinction has been drawn clearly for education in general, the focus will turn to mathematics education as the particular example chosen for examination in this thesis. It is hoped that the distinctions drawn at this point will be shown to reflect other distinctions already identified about the nature of mathematics, on the one hand, and the methodological theories of mathematics teaching, on the other.

In the conclusion, responsibilities will be laid out for any mathematics teacher to follow. It is hoped that the argument of the thesis will convince readers that these responsibilities are essential for anyone who has accepted that he wishes to introduce pupils, as part of their compulsory secondary education, to an understanding of mathematics. While it would be inappropriate for a philosophical thesis to identify syllabus detail or even prescribe the syllabus balance, it is believed here to be the philosopher's duty to identify the constituent approaches to mathematics education, if there is conflation and ignorance present. It would be to stand against a century of learning theory for a philosopher to claim that he can prescribe what content a mathematics syllabus must have, and for what age, in order that knowledge and understanding be achieved. However, some of what the adult mathematics teacher should know, to be able to formulate aims and objectives in mathematics education, will be prescribed: that is, he should be aware of the material identified in the first two parts of this thesis, which shows the breadth of usage of the term 'mathematics', and the differing perspectives from which mathematics may be taught. In other words, a framework for training mathematics educators is arrived at.

THE AIMS OF MATHEMATICS EDUCATION

In this section, there will be a brief explanation of the limited space given to consideration of the notion of 'education' itself, followed by a clarification of the terms 'intrinsic' and 'extrinsic' as used in discussions of the aims of education. This will lead to a discussion of the differing emphases given to intrinsic or extrinsic values by differing mathematics educators and how their positions are linked to their views of mathematics, mathematical understanding and approaches to mathematics teaching.

Education. Whatever sense one gives to this term, there would seem to be general approval of the necessary linking of education with an understanding of the main branches of knowledge.[1] Beyond this point, agreement seems to shatter into as many views as there are writers on education. Thus, any attempt to provide one further justified view of education would provide the writer with a task that would quickly outgrow the specific focus of the main body of this particular thesis. Rather than allow this to happen, this author will mention points about his implicit view of

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1. R. S. Peters, 'Aims of Education' in Philosophy of Education at one end of the spectrum of views on the concept of 'knowledge' to G. Esland, 'Teaching and Learning as the Organization of Knowledge' in Knowledge and Control, at the other. Whatever their differences over objectivity, they both accept that knowledge can be made available and can be used. This would seem to fit a base notion of 'knowledge and understanding' in an educational context.

education that may have influenced the emphasis of his present writings.

Firstly, it has been assumed that among the main branches of knowledge is some area that can be denoted as 'mathematics' without excessive ambiguity, beyond that clarified earlier in the thesis. It is taken for granted here that being educated should include among its achievements an understanding of mathematics.

Secondly, the author views 'education' rather as R. S. Peters views 'liberty', and 'equality'.[1] 'It constitutes a presumption. Reasons have to be given for interfering with people just as they have to be given for treating people differently.' (ibid., p. 180). Thus 'education' is to be seen as involving a presumption of 'optimism'. In an easily assimilated slogan this might be, 'every child can learn every thing'. The presumption is that, without good reasons for believing otherwise, 'educators must try to transmit knowledge with understanding, chosen by some further criteria, to all who lack the knowledge'. It is assumed that as a matter of fact, there is no adequate instrument by which one picks out those children who will be most enriched by, and will most enrich, any given area of knowledge. All learners are treated as capable of intellectual progress in all areas, unless conclusive evidence is shown of incapacity[2] being present. The practical consequence is that it is not thought right that a teacher in the state system should have the right to select whom, of those he normally claims to be able to teach, he is willing to teach. Admittedly, there are educational systems which reject these assumptions. They group learners according to the

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1. In chapters IV and VII of Ethics and Education, Peters places the responsibility for limiting equality and liberty on those wishing to introduce the constraints. It is this point about responsibility that is taken up here. Thus if a teacher wishes to exclude a child from his modern language class then he must be the one who has good reasons for so doing.

2. By 'incapacity' would be meant, the permanent inability to acquire knowledge in given areas, according to the evidence of scientific knowledge at that time.

belief that reliable statements can be made about educational outcomes, and that satisfying individual outcomes is overriding.

Thirdly, it is assumed that education is not just the transmission of a content to a group selected for good reasons but that the nature of the 'interference' at any one time can be justified. In other words, the author assumes that education involves 'interference' with children or parents or some other identifiable holders of rights on behalf of those to whom the branches of knowledge are to be transmitted. The most obvious way that the educator could justify this interference at the specific level of 'why is he teaching this to them now?' is by giving his aims and objectives. Thus the presumption made is that the educator could share his aims and objectives with pupils, parents or others concerned, as far as it is comprehensibly possible. The limitations should be recognised by the teacher through an on-going concern for psychology, sociology and philosophy, or more practically, the results of work in these areas as they touch the branch(es) he teaches.[1]

Thus the author sees education as the transmission of understanding of branches of knowledge, including mathematics, where the educator should assume that every child can gain that understanding, unless he can give reasons for this not being so; and also the educator should generally be able to identify publicly the aims and objectives underpinning what he transmits. In what follows, the discussion does not refer to 'education' exclusively in this particular form to which the author leans, but by giving the preferences now it is hoped that the orientation of this discussion may be the more readily appreciated.

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1. It might be taken as a natural consequence of this requirement that the teacher would think more carefully about <u>which</u> parts of syllabuses cannot be identified in terms of aims and objectives that are comprehensible to non-specialists, and why <u>they</u> are taught.

Aims in Education. While one teacher seeks to develop in his pupils a love of history, another seeks to develop skills so that his pupils are prepared for employment interviews. These two[1] teachers may be seen as emphasising different kinds of aims. The one stresses an aim intrinsic to education, and the other stresses an extrinsic aim. To the first teacher, 'a developed love of history' is part of what he means by 'being educated'. The aim is intrinsic to his conception of education. To the second teacher, 'possessing skills sufficient for an employment interview' is only an extrinsic aim of education. It is not a logical consequence of an appreciation of this teacher's conception of education, but the teacher may feel that such extrinsic aims are just as important for the future well-being of his pupils as members of society as intrinsic educational aims are. The first teacher may stress the worthwhile nature of the constituents of what he teaches, independently of consideration of them as means to ends external to education. He teaches history for its educational worthwhileness, while the other teacher may choose at the final analysis, the period 1900 to 1945, in preference to the period 1600 to 1685, if he thinks that an appreciation of the later period will be of more critical importance at an interview than an appreciation of the earlier period. This teacher considers extrinsic value to have the final say in his syllabus choice.

While this has provided an approximate notion of the distinction between intrinsic and extrinsic aims, there are finer divisions within each category that ought to be identified to prevent ambiguity in the consequent discussion. This clarification can be achieved most simply

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1. Although the passage talks of 'two teachers' it could be without contradiction, one teacher who holds both kinds of aims with considerable concern. The separation highlights the difference of meaning which is the important point here, between 'intrinsic' and 'extrinsic'. 'Extrinsic' is often replaced by the word 'instrumental' and there is no attempt to dispute such a substitution here.

by taking the notions 'intrinsic' and 'extrinsic' separately.

Intrinsic aim. The example of an intrinsic aim that was given above, 'a developed love of history', identifies a teacher's concern that his pupils will come to pursue history for its own sake, and that this will demonstrate in part that they are 'becoming educated'. In contrast, another teacher may be satisfied that his intrinsic aim has been achieved if his pupils acquire an adequate understanding of history. While the former teacher sees the pursuit of history as of overall importance, within a general notion of education in which the pursuit of truth is crucial, another teacher may be satisfied with the possession of historical facts. This teacher may see pursuit as just a means to an end, and the motivation which leads to the acquisition as inconsequential. Thus 'becoming educated' would have more to do with 'possessing the truth' than 'pursuing' it. Another teacher may see both these aims as adequate for he sees 'being creative' as the key aim. He wants his pupils to feel responsible for the development of this branch of knowledge, and thereby keen to try their hand at developing it, in case they are gifted in that way. Hence, the health of the branch of knowledge will be maintained, and more generally 'creativity' may be seen as the cornerstone of 'education'.

Thus, there is not just one sense of 'intrinsic aim' but a great many senses, only a few of which are hinted at here. Furthermore, a teacher may hold to several of the senses indicated here. He may want his pupils both to pursue and possess an understanding of history, and value both, in being thought of as 'educated'. This complexity and disparity of usage will be borne in mind, when the focus turns to considering the aims of mathematics education.

Extrinsic aim. In the same way that an intrinsic aim can have differing senses, so can an extrinsic aim. A vocational reason has already been mentioned. Another line would be a more general social one that pupils should study history in order to appreciate their cultural heritage and be attuned to its preservation.

Thus extrinsic aims may refer to long term effects on the individual, like his employment prospects or his independence, or they may refer to wider concerns like the society's heritage or preservation and/or love of the fatherland, and so on. There is a great diversity here too, and specific examples will be referred to in the discussion of the aims of mathematics education.

Before focusing down on mathematics, one note of caution may be worth giving. The line between identifying an aim as extrinsic rather than intrinsic may be less clear-cut than the examples so far given might seem to suggest. Particularly in the areas of language and mathematics difficulties can arise over their roles as 'handmaidens'. If a history teacher encourages his pupils to read Orwell and Hemingway to gain a better insight into the Spanish Civil War, is his objective extrinsic or intrinsic to his aim that they should pursue the study of history to become educated? Similarly with the physics teacher who encourages his pupils to take trigonometry seriously for work in optics, has he identified an extrinsic or an intrinsic aim? Certainly the study of literature in the one case, and the knowledge of trigonometry in the other case, will be used to develop the pupil's <u>intrinsic</u> valuing of history and physics respectively, but 'extrinsic' does not seem to be used here in the sense previously isolated. The physics example may be seen as even more difficult to untangle than the history example, for the pupil could not achieve a meaningful understanding of optics without knowledge of some mathematical theory of space. The mathematics is a logically necessary condition for achieving the aim of the physics teacher, that his pupils appreciate optics. There is a clear ambiguity here and its resolution would seem to be a matter of choice. What is important here is that awareness of the difficulty has been shown.[1]

Directing Mathematics Education. A professor of pure mathematics might say, 'Some members of each generation ought to have the opportunity to love symbols and their logical development'. A primary school teacher might reply, 'Mathematics education is a necessary part of learning how to run a home, and to make choices in one's life, independently of the advice of others'. While some will justify mathematics education on intrinsic grounds and others on extrinsic grounds, one cannot assume that educators will hold to a rigid consistency in their teaching programmes according to such emphases of value. The university professor may stress the intrinsic value of mathematics, but still proudly refer to the applicative value of matric algebra in traffic communication.[2] He may do this for motivational reasons, but, more likely, he is not completely blind, nor wishes his students to be, to the extrinsic value of some of mathematics. Similarly, a secondary school teacher, taking eleven year-olds, may generally talk of the value for life that mathematics has, but he takes care that pupils fully appreciate the elegance in the proof that a figure with n odd nodes, takes at least $\frac{1}{2}$n strokes to be drawn.[3] Clearly a commitment to one kind of justification of mathematics' place in education, is not a blank cheque for its use as the sole criterion of curriculum selection. Many other factors, many of them contingent, come in at this point. Obviously the teacher's job specifications, the lesson 'label' and the nature of mathematics itself, all constrain what is taught at any given time.

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1. This seems to be the basis of Heyting's justification of the value of mathematics as noted on p. 49 above. He suggests that mathematics helps one to think in certain ways. Generally one relates mathematics to 'thinking abstractly'. At this point it would be appropriate to stop and consider the matter psychologically, but at another time and place.

2. Telephone networks rely on matric algebra for drawing up plans of new systems.

3. Explanation of this element of traversable networks is described, for example, in S.M.P. Book B, Teacher's Guide, p. 295.

If one is given to understand that mathematics education is on-going, then this entails the presence of certain procedures, like derivability, which individuate mathematics. The complete set of such entailments will differ according to the teacher's view of the nature of mathematics itself. Furthermore, certain views will be seen to support, as a matter of fact, one form of justification of mathematics in education, rather than another. The Formalist view of mathematics, described in part 1 Chapter 2, falls readily into line with the intrinsic justification. Although Formalists argue about foundations, they agree that the central criterion of validity of new formally developed systems, is internal consistency, and nothing to do with extrinsic interpretation. In this sense, Formalists see themselves as <u>pure</u> scientists, and one may identify support for intrinsic justification in the concept, 'pure'. This philosophy may be readily contrasted with that of the Hypothesisors, who see any conception of 'truth' including a central criterion of external applicative value. Thus, mathematics is identified as necessarily an <u>applied</u> science, as all must be. This leads naturally, if not necessarily, to justifying the place of mathematics in education extrinsically.[1]

It is to be hoped that this last page has encapsulated the emphases indicated by the discussion of the nature of mathematics in part 1 of the thesis, and has set these beside typical, contingent perspectives about mathematics teaching, that had been considered in detail in part 2 of the thesis. The resulting synthesis is the recognition that arguments can exist, with support partially rooted in the nature of mathematics, for a justification of mathematics education, as intrinsically valuable, on the one side, and as instrumentally-orientated, on the other. Considering the extent to which these seemingly opposed positions are compatible

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1. Or, to see 'education' by definition, as involving the achievement of 'a better society' or other ideal state.

will be a key factor in the conclusion of this thesis.

Within any education system certain aims may be unrealistic.[1] Between a system in which mathematics was compulsory from five to sixteen and one in which, only arithmetic was allowed before the age of fourteen, one will expect to find strikingly different objectives of mathematics education at some points. In identifying aims and objectives, a mathematics teacher would be most unrealistic if he did not recognise political, social and other constraints, particularly if they were of the kind just identified. In addition, he will have his sights adjusted differently, if he leans to the ultimate aim of children coming to value mathematics for its own sake, rather than towards their always being concerned to know the immediate 'pay-off' of what they are learning.

It is no part of a philosophical analysis to present a sample of aims and objectives. It is the job of a philosophical analysis to pinpoint inconsistencies that will arise in selection, if certain principles are not recognised,[2] and also to indicate that what look like inconsistencies are not logically so. Consider a teacher who aims to develop manipulative skills in real numbers, but has no specific objectives like dealing with accounts, calculating electricity bills, etc. From the views presented in this thesis, it should be clear to see that this 'anti-common sense' position can have a consistent foundation. The teacher may not simply be pig-headed, but is presenting a strictly formalist view of mathematics, through the games perspective. He could fear that pupils will be content with the applicative value of mathematics if that alternative is presented, while he believes the true value of mathematics lies in its intrinsic value. Hopefully, this is

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1. This has become even more evident with the recently increasing emphasis upon accountability. No attempt has been made here to argue that the teacher must be autonomous in the formulation of his aims.

2. Such principles are discussed below, pp. 264-67.

the reason. The fear is that the teacher is aware that his pure mathematics professor at university had this attitude, and held it on rational grounds, but that the teacher himself holds this view only because of emotions kindled in him by the professor. The professor is knowingly in whole-hearted support of the view that mathematics is to be loved for its own sake, but his protégé in the secondary school may appreciate and implement the attitude, yet lack the philosophical understanding that gives it rational foundations. It is equally tragic if a college-trained mathematics teacher talks continually of 'discovery' and attempts to implement a Nuffield-style approach, but the teacher lacks any adequate knowledge and understanding of this as a scientific perspective.

It is to be hoped that some of these complexities, underpinning the various movements of mathematics education have been clarified, or at least, identified. If the complexities are there, then it is not so surprising that the average mathematics teacher is defensive when he is asked to justify his own standpoint, or to inspire pupils to follow the 'cause' he seems to represent, as his professor had inspired him. The 'causes' are many, and the nuances so technical, that an appreciation of distinctions thought critical by the followers is completely out of the reach of secondary, let alone, primary pupils. An additional cause for concern is that teachers too, are unaware of the complexities. Consider the complex nature of the position held by a teacher who views the nature of mathematics as the totally abstract 'Science of Formal Systems', while being fully aware that his present class are at a concrete level of cognitive development. As a good teacher he may well stress the 'care' to be taken in the connecting argument of a set of concrete operations, (e.g. when one considers what numbers of points it could require Chelsea, Newcastle United and West Ham United to gain promotion from the Second Division). This teacher highlights rigour at his pupils' conceptual level. Unfortunately, many teachers are less rigorous when they think about their teaching than when they think about their subject. The result is an attempt to provide illumination of mathematics through formal manipulations before pupils are intellectually ready.[1]

Thus the aims of various mathematics teachers have been considered, and the result has been a great diversity of models of mathematics educators. These models have been predominately orientated towards the view that the nature of mathematics rests in some form of 'formalisation'. They have also for the most part emphasised the intrinsic value of mathematics in education.

The diversity of types of intrinsic aims have been considered earlier, p. 240. While the professor imagined on p. 242 above stresses 'pursuit of truth', the teacher on p. 245 seems more concerned with the 'possession of knowledge', as he 'aims to develop manipulative skills in real numbers'. The critical point is that this diversity exists at several levels[2] and a mathematics educator can consistently choose 'his own mixture', if he has the kind of knowledge expected of a philosopher, that will prevent his falling into logical pits. Thus the teacher cannot claim to emphasise the extrinsic values of mathematics, if the only view of mathematics his pupils receive is linked to 'loving mathematics for its own sake'.

Contrasting views of mathematics educators have been given in these fictional models both in the previous few pages and earlier in the thesis. It is crucial to the argument of this thesis that alternative standpoints are internally consistent and should be taken into account. This is not just interesting theoretically, for the main situation in which aims

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1. A recent discussion of this point is found in the Introduction to <u>Notes on mathematics for children</u>, C.U.P., 1977, pp. ix to xvi.
2. There is diversity about the nature of mathematics identified in part 1 of this thesis, and similarly about teaching perspectives (part 2), and also about aims and objectives as identified in the present discussion.

and objectives of mathematics education will be discussed is in the department meeting of a secondary school mathematics team, and such meetings would be either totally 'pragmatic' or lacking in communication, if some members of the team held that there was only one correct conception of the pupil becoming mathematically educated, and any other was necessarily invalid. While there may be common features and there may also be some logically impossible combinations, the fundamental view presented here is that there are possible alternatives.

One of the features that has been presumed to be common to all the various kinds of aims of mathematics teaching has been that 'understanding' is involved. As yet, there has been no attempt to clarify what is meant by 'understanding',[1] particularly in the specialist area of mathematics. It would seem appropriate to consider in the light of the present discussion, 'Is"mathematical understanding" ambiguous?'
It would be hoped that this will lead to a reinforcement of the central commitment to alternatives.

<u>Mathematical Understanding</u>. In part 2, Chapter 10, two conceptions of mathematics were identified in relation to the Technology Perspective. (1) A <u>weak</u> form, in which mathematics could be said to satisfy the perspective if, as a matter of fact, it could be shown to have connections with other areas, and to have an internal interconnecting structure - as matrices generalise principles identified in the solution of simultaneous equations. (2) A <u>strong</u> form, in which mathematics is held necessarily to have both internal and external connections. These conceptions parallel differing views of mathematical understanding which will be presented now.

In the <u>weak</u> form, mathematical understanding consists of understanding

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1. On pp. 150-153 above, there was an attempt to indicate some qualities a pupil would show if he 'understood' what he had been taught.

connections within mathematics related to the particular proposition that one claims to understand. Thus, one may rightly claim to understand that the angle at the centre of a circle is twice the angle at the circumference,[1] if one can derive it from axioms and can go on from the proof to demonstrate that any cyclic right-angled triangle has a diameter as its hypotenuse. This view is readily compatible with Wittgenstein's linking of 'understanding' to 'use', as discussed on p of this thesis. No claims or conditions are made for 'use' outside the formal system.

In the <u>strong</u> form, mathematical understanding involves at every point consideration of the possibility of knowledge about the given proposition including its use in other areas. Thus an adequate understanding of the theorem mentioned for the 'weak form', would necessarily involve some knowledge of its external use, for example, in the positioning of cameras at an athletics meeting to maximise coverage.[2] This is Ormell's position. He feels that one is not being sufficiently demanding if one simply requires an understanding of the applicability of mathematics as a whole, rather than making applicability to 'reality' a key feature of the understanding of any mathematical proposition.

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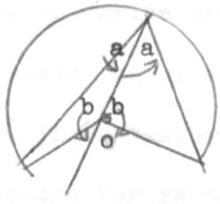
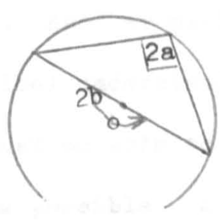
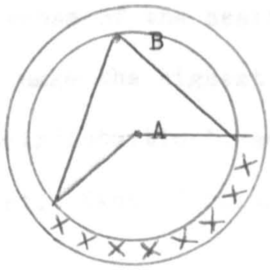
1. a) $2(2a) = 2b$



b) If $(2a) = 90$ degrees, then $2b = 180$ degrees.



2. The cameraman at B needs only half the camera angle of the cameraman at A, and so positioning cameramen at the circumference maximises use over those at the centre. (Example suggested by C. P. Ormell)



x Competitors in a race.

The arguments presented in part 1 Chapter 4, and part 2 Chapter 10 have been that the Hypothesisors see the use of a mathematical proposition not as a bonus, but as an essential part of its meaning. Thus, an essential part of the strong form of mathematical understanding is demonstrating that one knows the kinds of place where external applicability of a potential model may be realised.

While not wishing to require the strong form, one may wish to go beyond the weak form in requiring the person with understanding to have some appreciation of how mathematics, 'taken as a whole relates to things that are different from it'. This is the standpoint found in the work of Stephen Brown who encapsulates his view as 'to understand X internally is to see connections within X itself: to understand X externally is to see how X taken as a whole relates to things that are different from it'. (Mathematics Teaching, No. 69, December, 1974). A pupil may have 'weak understanding' of matrices but no idea of what separates that area of study from that on circuits in another area designated 'physics'.

It could be useful to consider how questions about mathematical understanding may enter the discussion going on in the department meeting mentioned above. Differing teachers may stress different features that they consider critical to a pupil's development of mathematical understanding. The head of department may wish to stress the objective that students are able to derive and appreciate formal proofs. Another member of the team insists that a stronger notion of mathematical understanding is required in contemporary society. The pupils must be able to see external applications for as much of their mathematics as possible. A third member of the team fears that following the views of the other two will be at the expense of the healthy growth of mathematics itself. This teacher wishes to make the highest priority the encouraging of pupils to take risks, use their'intuition', so that future creative mathematicians should not be lost. Each of these positions will be considered in turn:

1) Formalisation. The head of department is satisfied with the weak sense of understanding, and follows in the direction of the past fifteen years, in which the stress has been upon formalisation. He may even accept that this has been at some cost to the level of arithmetic competence of his pupils, but has led to improved understanding.

However, there do seem to be some other weaknesses in this position. As Poincaré said, about three-quarters of a century ago, 'in becoming rigorous, mathematical science takes a character so artificial as to strike everyone; it forgets its historical origins; we see how questions can be answered, we no longer see how and why they are put' (The Value of Science, p. 217).[1] It is algebra that is insisted upon in 'modern mathematics', and it is algebra which is most formalisable, having the richest syntax. The result is that children are required to study most what has no direct empirical representation. '2 + 2 = 4' can be represented by apples, and the triangle by a piece of paper, but how can one concretely represent 'a o b = b o a', that commutivity holds for some algebras and not others? Algebras stand one further step away from the everyday, and one explains the commutivity, by comparing '3 + 1 = 1 + 3' with '3 - 1 $\neq$ 1 - 3', or 'the cat is beside the dog' means the same as 'the dog is beside the cat' but 'the cat is under the dog' does not mean the same as 'the dog is under the cat'. In other words, if one considers arithmetic, geometry and algebra as languages, then algebra is least like ordinary language which has semantic clarity and syntactic exceptions. There is a natural hierarchy from the everyday to the algebraic, and one ought not to be surprised to find sense in mathematics education delaying the algebraic. This is not a new argument, but is

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1. Meserve quotes Poincaré in his article, 'Geometry as a Gateway to Mathematics', in Developments in Mathematical Education, p. 251. Like Thom, Meserve indicates that geometry combines an intuitive clarity with applicability which makes it a strong contender for the role of 'gateway to mathematics'.

found in much the same form in René Thom's article, 'Modern Mathematics: Does it Exist?' (loc. cit., pp. 194ff). The stress on formalisation and algebras led education to forget that for most people for most of their lives mathematics means numbers and their manipulations. To start with set theory in the secondary school, or even earlier, is to confuse the pupils unnecessarily. Thus the brave words of S.M.P's Director, Brian Thwaites, in his 1967-68 Annual Report, (The First Ten Years, p. 176) that 'The familiar "Oh Sir, what's the point of this?" should now be as obsolete as totting should have been a generation ago' have a strange ring, when considered alongside S.M.P's own Supplementary booklets in the nineteen seventies, concentrating on the very manipulative skills that were rejected in S.M.P. Book A. There one reads that the purpose of the section on fractions 'is to remind the pupils of the meaning of fractions; it is not concerned with complicated techniques for combining them'.

Seeing the public meaning of mathematics, as can so readily occur in geometry, and having confidence that one has got things right, as occurs in arithmetic, are both essential for a child to become a 'fluent speaker of mathematics'. These contingent factors were missed by those who were keen on formalisation at any price. The argument is not for all arithmetic and geometry, and no algebras, but that formal derivability has only a limited role in mathematics education.

2) External applications. For the teacher with belief in the strong form of mathematical understanding, there is a great intellectual drive to insist that pupils, provided they can manage it, tackle syllabuses that either take cognisance of external applications or are adaptable to such an approach. Clearly, this adds weight to what is required of pupils. The demands are even greater if one takes cognisance also of the arguments in part 2 Chapters 9 and 10 above, that 'applicability'

is an 'interactive' process and so one must have understanding of the area to which one applies the mathematics, as well as the mathematics itself. The implication would seem to be that if the syllabus content is unchanged when one adds this extra dimension then there will be increased strain on the pupils. This strain can be partly ameliorated if one accepts the point made in part 2 Chapter 10, p. 197, that '"Rigour" is tied to "caring" rather than exclusively to an obedience of axiomatic sequences.' The teacher may be satisfied with a child at a particular conceptual level having visual evidence that the angles of a triangle add up to a straight line,[1] for he wishes to concentrate on its usefulness as quickly as possible, and there is nothing contentious about the rigour of the existent formal proofs. One reduces the demand on the pupil by not requiring him to be able formally to derive every proposition he applies, but only those which may require modification to facilitate successful application.

3) <u>Intuition</u>. This last point well exemplifies the worry of another teacher who wants to encourage the pupils to have confidence to be 'creative' in their study of mathematics. Like the teacher who is keen on external applications, he may have to predetermine which are the important formal proofs to prevent his pupils being overloaded. Thus the pupils are directed to what is to be 'challenging'.

Similarly, formalisation may be stressed to the extent that pupils come to believe that, given logical reasoning, everything in mathematics can be worked out, but in practice the routes are rarely so clearly

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1. He simply requires the pupil to tear off the corners of a triangle and rearranges them to form a straight edge.

The empirical exercise provides adequate assurance in this case.

demarcated by logic alone. Even if mathematics is nothing but the science of formal systems, the point made in part 2 Chapter 9, p. 167 is worth repeating here, that children are given a false understanding of mathematics, if they are only allowed to see it as 'a tidy system'. Developments by real mathematicians are untidy, but the pupil is given the impression that formal derivability is all that one needs to know in order to do mathematics. As Hadamard is quoted as saying, 'the object of mathematical rigour is to sanction and legitimate the conquests of intuition, and there never was any other object for it' (Polya, 'As I read them' in Developments in Mathematical Education, p. 78). The danger of some formalisation approaches has been that children have neither been given signs that there is a place for intuition, nor sufficient confidence in the probability of repeated success to use their own intuition;[1] if, indeed, they are told of its existence.

It can be argued further, that stressing the ability to derive and appreciate proofs, as the central objective of mathematics teaching, overshadows a key feature of mathematics, the clarity of failure. A pupil's confidence can be built up if he recognises that mathematics may develop through picking up the interesting features of the mistakes one makes: then one will be less reluctant to rely at times on one's intuition, and not feel frustrated if results do not 'drop out' tidily. This feature of encouraging pupils to help develop mathematics may be called:

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1. 'The development of self-confidence and courage in the student would certainly appear to encourage intuitive thinking. This requires a willingness to make mistakes and one who is insecure or who rather is afraid to enter the realm of insecurity may be unwilling to make such mistakes.' (J. W. Oliver, 'The Role of Intuition' in L. R. Chapman (Ed.), The Process of Learning Mathematics, 1972, p. 65)

Learning from Failure and Untidiness. Early in their schooling, children are taught to check their answers to sums. Given a subtraction sum, they are expected to check the correctness of their answer by addition. This increases the pupil's independence and confidence, and also reinforces the message that addition and subtraction are inverse operations. Students of mathematics are bred on this clarity of failure. To their advantage, pupils are taught methods by which they can identify their mistakes for themselves, but[1] this approach also leads pupils to expect that every answer given in mathematics can be given a clear tick, or cross. Hopefully, teachers are increasingly stressing the positive side of this peculiarity and playing down the negative side. If pupils are made to realise that they can learn through their mistakes, and that mathematicians in general learn through their self-corrections, then this is one way in which pupils will come to appreciate that mathematics can be 'untidy'. This untidiness has been viewed as part of the successful development of mathematics. It is not the only subject in which people benefit from their mistakes but is the only one[2] in which the mistakes can be identified by the person who makes them, through the correct displaying of related procedures, like inverse operations.

A further peculiar feature of mathematics learning, that is only open to a minority of other areas,[3] is that stressed by Wittgenstein. He pointed out that someone who learned a new proof in mathematics was simultaneously doing some new mathematics. Learning and doing are one.

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1. A question of value is indicated here and a preference is shown. It is assumed that most children view the clarity of failure in the way described. Furthermore, it is prescribed that teachers ought to try to minimise the number of pupils who associate mathematics with failure.

2. This is true only if the subject identified is the science of all formal systems, for clearly a propositional calculus has this property of self-correction.

3. As far as the author knows, there are only claims that ordinary languages have this feature of simultaneous 'learning and doing' but they are not self-correcting.

This peculiarity highlights the inappropriateness of seeing mathematics as a game, as was presented in part 2 Chapter 8. To learn a new opening in chess, is not to play chess. The learning is logically separable from the playing of the game, but this is not so in mathematics. This peculiarity does not have implications for teaching approaches only, for differing philosophies of mathematics will interpret this feature in different ways. The Formalist will see the simultaneity of learning and doing, as relevant only to the learning of fact-free symbols within a given system, while the Hypothesisor will see any extension of mathematics as potentially referring outside of the science of formal systems itself, as a further 'potential model'. The Hypothesisor puts the added responsibility upon the mathematician to find out what external implications follow from any changes he makes to the formal system internally.

Whatever one's philosophical standpoint a mathematics teacher ought to recognise the peculiar place that correctness plays in the learning of mathematics. The child who is able to correct his own work is also able to correct anybody else's work. In this sense, being able to understand one's own mistakes entails being able to understand what one has done wrong. This leads to a further point about 'mathematical understanding': someone who can rectify as well as recognise his errors in a given area has the more developed understanding, at least in that respect.

Thus an appreciation of how to correct mathematics is itself an indication of the level of mathematical understanding achieved. Two points can now be made in response to the question given on p. 247 above 'Is "mathematical understanding" ambiguous?'

1. It is ambiguous. Different teachers do mean different things by it. BUT
2. There would seem to be certain commonly agreed features if one wishes to achieve particular intrinsic aims for the mathematical education of one's pupils. Thus any teacher concerned about the health of that branch of knowledge called 'mathematics' needs to encourage his pupils' <u>intuition</u>

and that is facilitated by highlighting the positive value of appreciating error and untidiness as sources of mathematical development. Similarly there is agreement that to achieve the aim of possessing knowledge of mathematics with understanding will involve to a greater or lesser extent the appreciation of the formalised nature of mathematics. However, 'external applicability', which is seen by this author as a critical feature of mathematical understanding, is not considered important by every secondary school mathematics department; nor is it claimed here that every department must make external applicability a key aim. However, the following discussion will highlight the danger of being satisfied with the weak form of mathematical understanding.

Teaching mathematics: its unity and isolationism. While it has just been admitted that teachers cannot be forced to take the view to which this author leans, it would be a sign of weakness if the full weight of argument is not presented for greater concern for external applicability. The opportunity will be taken here to spell out what is seen as one further critical danger for all education that results from the mathematics teacher's overriding concern for the 'internal health' of his subject. The argument given here does not denounce the validity of such concern but questions whether it is the most appropriate central aim in the educating of secondary age pupils, and whether other, less specialist aims, are more appropriate for the main body of pupils who study mathematics during their compulsory schooling.

Given any group of mathematics teachers, there will be division among their chosen objectives, but there will be unity also. Over the last five years, teachers have taken to heart remarks made about the radical changes made to mathematics education in the previous decade. In Mathematics Teaching, No. 73, December 1975, Dunning-Davies brought the charge, that there was 'a pre-occupation with jargon and abstract mathematical structure at the expense of much needed manipulative skill'. This may be seen as a counter to the kinds of exhortations to mathematics teachers that had been preached by projects like S.M.P., in whose first report, one reads that, 'Almost any required index of manipulative skill, within reason, could be achieved at sufficient cost to the rest of the pupil's mathematical education'. By these quotations one can readily identify the opposing pressures put on mathematics teachers for the last fifteen years, at least. They were first encouraged to drop arithmetic for structure, and then to reduce structure for manipulative skill.

A philosophical thesis is not the place to draw conclusions about the psychological validity of one view or another, in terms of its proven efficiency in educating pupils mathematically. However the philosopher can make it clearer what can be meant by 'manipulative skills', 'structures', and so forth. He can also indicate where ultimate goals are inconsistent with initial presuppositions. The phrase 'manipulative skill' may be used to imply any competence, from one that is solely the result of 'rote-learning' or any other non-manipulative response control,[1] to some other skill which cannot be achieved competently unless the learner recognises the rules he is to follow. Given the request for an answer to 'three treble tops' at darts, three different people may have differing levels of understanding, hidden in their correct responses.

1) Johnny automatically answers '180'. He has some understanding of subtraction and can work out '501 - 180', but all his multiplications are automatic. He knows by heart all the dart-board combinations from double one to three treble tops. Taken away from the dartboard and Johnny would be at a loss with '3 x 3', let alone '3 x 3 x 20'. He has no formal skills of multiplication.

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1. A child learns to respond 'three' to the shape '3', but this may indicate nothing of his knowledge of arithmetic, but only of his powers of imitation.

2) Joe can respond '180' to both 'three treble tops' and to '3 x 3 x 20'. Joe can work out '4 x 2 x 12' even though it does not appear on the dart-board. He has some recognition of 'written arithmetic' which is lacking in Johnny. Joe has a rote-learned knowledge of all number bonds from 1 x 1 to 12 x 20. He has 'arithmetic fluency' in the sense that as a milk boy he can work out in a book, for the milkman, what each household has spent that day, week, etc. The milkman does not require Joe to have heard of associativity or commutivity but he does work according to such rules. In this sense and in the sense that he applies his arithmetic, Joe could be said to have some understanding of arithmetic.

3) Jill has an internal understanding of arithmetic in the finite domain up to '12 x 20', for she knows that '3 x (3 x 20) = (3 x 3) x 20' and that '3 x (3 x 20) = (3 x 20) + (3 x 20) + (3 x 20). She appreciates the rules of associativity and commutivity for this domain, and that multiplication is a recursive form of multiplication. She is certainly on the first rung of the ladder of mathematical understanding, the abstract activity.

When there is a discussion of weaknesses found in young people's manipulative skills, it is essential that all parties know what is being identified. No one ought to be satisfied with Johnny's highly restricted state,[1] but the accusations of Dunning-Davies and others seem to be directed at the Jills of this world, who may lack 'arithmetic fluency' although they can explain the nature of the activities they are asked to carry out. To claim confidently that Jill has mathematical understanding in the modified sense,[2] identified on p. 249 above, at least, the teacher

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1. The automatic response is acceptable if it 'is seen as a quick way of summarising a process of thinking rather than the acceptance of an authoritarian statement' (M. Brearley, Number in the Primary School, Froebel, 1960).

2. In that sense of mathematical understanding, Jill is required to show some appreciation of connections internal to the area of study, and also appreciation of how that area of study relates to other areas, external to it.

must be able to show that Jill is aware of the applicative value of arithmetic as well as the nature of such interconnections as those between addition and multiplication. In other words, the desire is that the divide between Joe and Jill should be removed and that a minimum level of 'manipulative skill' should also indicate the moderate form of mathematical understanding, at least. It would be just as unacceptable for teachers to strike out for 'Joes' at the expense of 'Jills' just because the man-in-the-street is more concerned with applications than understanding rules, as it has been to ignore 'Joe' for 'Jill' in recent decades. It is to be hoped that people will come to accept the broader notion that combines both sets of skills, and not be swayed in one direction by public opinion, and in the other, by editorial comments that announce that '...mathematicians are among the last people to need arithmetic' (Mathematics Teaching, No. 73).

What seems to bedevil this rational compromise is the belief that Jill's understanding is in any sense inadequate, unless she appreciates that arithmetic is embedded in set theory. She may understand that multiplication is connected logically to addition, but not that the integers can be derived formally from set theory. John Williams has said, 'unless the ELEMENTARY structures have been grasped, those that incorporate them will not be intelligible' ('Some Peculiarities of Calculative Thinking' in Learning and the Nature of Mathematics, p. 180). A house of cards models this standpoint and a child who has missed a foundation card will soon find his house topples down. This is the view that mathematics is so rigorously formalisable that one must start with the ground rules. Given the games perspective discussed in part 2 Chapter 8, one should not be surprised that there is such a view. One would not be surprised to be told that it is stupid to try and play draughts, if one has no knowledge of how many squares each piece may move, nor upon what colours on the board. While it is consistent, if one holds that mathematics is one big game, to require the child to start with the ground rules, the agonies that result are surely good reasons for most of us shying away from this, as a pedagogic approach at least.

A warning note is being signalled. Over-emphasis on the coherence of mathematics leads as a matter of fact to impossible demands being placed on the child. Although no question is raised for the philosopher of mathematics, a logical point is identified for the philosopher of mathematics education. Anyone who presents as realistic a programme of mathematics education which is totally inconsistent with all agreed knowledge in educational psychology, is committed to as much a logical absurdity, as someone who claims that the square root of 2 is rational. It is more than strange for someone to admit that there is a lot of sense in the work of Piaget, Dienes, Bruner et al., but still to go on wanting children to begin with the most abstract mathematical structures. Unfortunately, this does seem to have occurred and helps to explain the placing of infinite cardinals in the work for eleven year-olds in series like Learning Mathematics and the Manchester Mathematics Group. Coherence must be tempered by a concern for applicability and realism, if the task of getting pupils on the inside of mathematics is to be a generally reasonable one. The objective of mathematics education being supported here is that, in this area, 'the achievement of manipulative skills, with understanding, must include that conception of understanding, identified in its strong form,[1] on p. 248 above'.

Some educators may argue that this objective can only be satisfied if one takes some form of the inter-disciplinary approach outlined in part 2 Chapter 11. Built in to this objective is the imparting of knowledge in other areas besides mathematics and this would be achieved most

1. The strong form of the concept of mathematical understanding, outlined on p. 248 above, involves a mathematical proposition being understood only if the person knows how to use and derive it in mathematics, and has some awareness of its potential for external applicability.

efficiently if there was 'interaction' among teachers as well as between the disciplines they teach.

Until recently, the weight of mathematics education has been 'isolationist'. Mathematics teachers have been reluctant to participate in the growth industry of inter-disciplinary projects and group C.S.E's. Peculiarities of the subject have been seen as justifying the separation of mathematics and its study from everything else. Part of the argument has been the claim that the systematic nature of mathematics will not come across to pupils so clearly if it is tainted by external applicability. If children are to see mathematics as systematic, and if there is no adequate model of mathematics, then the argument seems to be that children in secondary schools ought to be able to look back on their mathematics career and perceive the visible system, with the peculiarities that have been identified.[1] The argument is essentially value-laden even though it carries the support of almost all great pure mathematicians this century. Hardy stressed the importance of extrinsic uselessness of his work. His pleasure lay totally in the doing and the aesthetic experience that often resulted. Communication to others was inessential, the doing was all. Thus, one has a picture of the isolated mathematician, active in mind and suspended in time. This is comparable to someone doing The Times crossword on an endless journey to their office. Hardy was willing to discuss his work with like-minded people but he refused to accept any responsibility for the applications that his work might have.

While a genius may be allowed to hold to such isolationism, it would be senseless for generations of ordinary children to be told that isolationism is an essential part of the wonder of mathematics and that it is not to be studied in the light of society and its problems. Added

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1. The argument of part 2 Chapter 11 was that this does not logically preclude its Inter-disciplinary Study (See particularly the Conclusion, pp. 233-4).

to all the mathematics teachers who genuinely believe in this isolationism must be added those who have gone with it for 'a quiet life'. Such a teacher is concerned with achieving the maximum measurable set of behaviourable objectives with the minimum level of disciplinary disturbances, and never takes seriously the choice of a more open standpoint.

The isolationism identified in this part of the thesis does not rest explicitly upon views of the nature of mathematics, for this line of argument was the key to the discussion in part 2 Chapter 11.[1] Rather, the argument of the isolationist mathematics teacher is that learning mathematics involves peculiar features like 'self-correction' that are not found, as a matter of fact, in other areas. The hidden consequences of this have included reliance upon a paradigmatic form of problem-solving. At all levels up to 'A' level, ramified forms of this method have taken people through their examinations.[2] This habit learning' is not exclusive to mathematics but the 'isolationism' has reduced the probability of others being aware that the kind of 'knowledge and understanding' identified in this part of the thesis is not being achieved or even considered as an essential objective. Teachers slip into believing that 'getting them right is synonymous with understanding'. Furthermore, there would seem to be a worryingly short set of steps, to 'following algorithms blindly', from the formalist, met in part 1 Chapter 2, who stressed the rule-governed nature of mathematics. The formalist mathematics teacher, believing that struggling blindly is part of learning to love mathematics in the end, may well argue that giving pupils extrinsic reasons for learning some

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1. The argument presented on p. 218 above is that each philosophy of mathematics identifies criteria by which mathematics is shown to be 'discrete' and is in that sense, isolatable area of knowledge.

2. A child is taught to treat any problem of the form, 'I have some..., and then get --- more..., and I have --- .... all together. How many did I have at first?' By using a given flow chart, verbalised as, 'Take the last mentioned number, subtract the penultimate one and write down the remainder as your answer', the child is given a method, but not understanding.

element of mathematics now will only lessen their commitment to mathematics for its own sake.[1] Certainly this teacher is not an educator, in the sense identified in this thesis, and can have no educational defence for his approach, if he claims to be educating his pupils by 'blind obedience'.[2] The alternative discussed in part 2 Chapter 8 was the teacher who puts off mathematics education until after the pupils can grasp the conceptual demands made by the abstract study of 'the science of formal systems'. While valid, it is not easy to imagine many teachers admitting that their subject is 'pseudo-mathematics'. For a teacher with a genuine commitment to Formalism, this would provide the benefit that 'isolationism' would be unnecessary, and could be delayed. It is to be hoped that the emphasis of this section has been seen to be the breaking down of walls and the building of bridges among teachers of different subjects, as well as in the minds of pupils, as they employ the techniques of one subject and then another.

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EDUCATING MATHEMATICS TEACHERS

This whole thesis has been aiming at revealing a part of the essential equipment of a mathematics teacher. The previous section has discussed teachers' aims, expectations and even models of their mathematically educated pupils, but no attempt has been made to direct the analysis inwards at the teacher himself. Now, therefore, the question is asked directly: 'What essential equipment for a mathematics teacher can this analysis identify?' In order to answer this question one would need some idea of the context of this discussion, and a context of some

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1. This is analogous to the piano teacher who demands nothing but exercises from his pupils, never allowing them to play complete pieces, at their level.

2. He could claim to be laying the preconditions for their becoming mathematically educated. This problem was discussed previously in part 2 Chapter 8, pp. 150-154 , in particular.

kind has been identified in the points made about 'education' itself on pp.236-38 above. The conclusion drawn was that education is to be seen at least as, 'the transmission of understanding of branches of knowledge, including mathematics'. If teachers are to transmit an understanding of mathematics then it seems to follow that they need types of 'equipment'. These could be put crudely as, 'equipment on the subject to be taught', 'equipment on teaching the subject', and finally, 'equipment on aims of teaching the subject'. Obviously, a practising teacher will not see these as discrete areas, and throughout this thesis points of interaction and overlap have been highlighted. This does not deny the possibility that treating each area separately here will not benefit the clarity of the overall argument of the thesis. Thus the three types of 'equipment' will be taken in the order suggested:

1) Knowledge and understanding of, as well as competence in, mathematics

In the first part of this thesis it was shown that 'mathematics' itself has a complex nature. It was argued that there are at least four internally consistent ways of interpreting the structure of mathematics. These were identified as the Logicist, Formalist, Intuitionist and Hypothesisor philosophies of mathematics. In order to claim a fuller appreciation of the nature of the subject that they will teach, it would seem reasonable to require future mathematics teachers to have some background experience of these viewpoints, in addition, where necessary, to the knowledge of the subject acquired in their non-professional training.[1]

2) Knowledge and understanding of differing teaching approaches to mathematics

In the light of all that has been written in the second part of this thesis, this may seem like a very broad requirement, but nevertheless it is surely a reasonable consequence of the basic notion of 'education'

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1. If their previous or concurrent study of mathematics lacks study of the nature of mathematics this should be compensated for in training.

considered above. This included a reference to 'transmission' and so it is reasonable to assume that a future mathematics teacher would wish to be equipped with knowledge of differing approaches to mathematics teaching. At least some of these approaches were identified in part 2 of this thesis, and the future teacher would be in a stronger position to claim an understanding of these approaches if he was also aware of the kinds of links that have been mentioned in the thesis between certain approaches and certain views of the nature of mathematics itself. Thus in part 2 Chapter 8, the harmony of the games approach with Formalist views of mathematics was picked out, and in Chapter 10 similar harmony was found between the technology approach and the philosophical standpoint of Hypothesisors. Four discrete approaches have been identified, as well as a fifth derivative one. These are 1) the Art approach, 2) the Game approach, 3) the Science approach, and 4) the Technology approach, and teaching mathematics through Inter-disciplinary Enquiry.

3) <u>Knowledge and understanding of aims of mathematics education</u>

While this area may require a key position in the training sequence of future mathematics teachers, it has been placed last here and in the thesis as a whole, because it is the most controversial area to tackle. Underlying it are questions of the following kind:

'Why teach mathematics to them?', and the slightly less contentious,

'Why teach that part of mathematics to them?'.

An educator of mathematics teachers can equip his students to understand the notions of educational planning, and even to appreciate the different programmes[1] of mathematics education, that have, do and could exist, but ultimately the teacher must decide for himself how to answer these kinds of question. The philosopher can help him to pick out likely logical

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1. Programmes devised under a great variety of constraints, including greater and lesser degrees of teacher autonomy and accountability.

pitfalls, but cannot make the choice of values for him. Thus this thesis could end here, but on p. 237 above, the author admitted that he hoped that others so interested, would include in their understanding of the notion of 'education', more than the transmission of understanding of the main branches of knowledge, but also

a) a presumption of 'optimism', and

b) a presumption that teachers can share their aims with pupils, parents or others concerned.

It would be clearer to consider each of these additional factors in turn:

a) A presumption of 'optimism'. If the educator of mathematics teachers wishes his students to be aware of the consequences of this presumption then he would expect them to understand these consequences to be that:

1) Everyone can enlarge their knowledge of mathematics.

2) Everyone should have the opportunity to appreciate that mathematics has this characteristic of enlargement (e.g. $1 + 1 = 2$; $2 + 1 = 3$; $3 + 1 = 4$; $4 + \ldots$).

3) Everyone should have the opportunity to achieve some understanding[1] of mathematics.

A generalised consequence is that,

4) Every teacher ought to assume that his aims of mathematics education refer to every one of his pupils unless he can give good reasons why they should not.

This leads to the second presumption which requires mathematics teachers to give, as far as possible, an account of why they teach what and how they teach it.

b) A presumption of 'sharing of aims'. If a teacher constantly rejects the right of pupils, parents and others concerned to be given reasons, besides that of saying 'this is necessary for you to become educated',

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1. Obviously this would involve differing demands according to the sense of 'understanding' taken.

then he would not be satisfying the notion of educator that is presumed here. The teacher need not be required to justify his every move, but he must be required sometimes to be prepared to tell his pupils or others the purpose of what is to be learned, and why it is being taught in a particular way rather than in another.[1] In this sense, a teacher has a responsibility to have the knowledge and understanding required, so that he is able to indicate to the pupils the relevance of what he teaches to them individually, or to society as a whole.

Thus, three aims have been identified for any educator of future mathematics teachers: The student will acquire

1) knowledge and understanding of, as well as competence in, mathematics;

2) knowledge and understanding of the differing teaching approaches to mathematics;

3) knowledge and understanding of aims of mathematics education.

Admittedly the three aims given briefly here are not very informative by themselves, but it is hoped that, supported by the thesis as a whole, they would provide an essential part of a framework for training.

CONCLUSION

In the first part of the thesis, four philosophies of mathematics were presented. While some preference has been shown for one movement, that of the Hypothesisors, rather than the others, the critical conclusion drawn is that there are several such movements, each one of which is internally consistent. The ideas that they present have influenced mathematics education since 1944. Although the formalising pressures of logicism and formalism are to the fore in most modern mathematics series like Learning Mathematics, S.M.P., and others, intuitionism has reminded some mathematics educators of the limitations of the written

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1. This is setting aside motivational support for such a presumption, which is probably very strong.

form. This comes across particularly clearly in A.T.M's Notes on mathematics for children (C.U.P. 1977, pp. 224-30). The following introduction to the section entitled 'Creative Ignorance' should highlight this approach, as it questions formalisation pursued at any cost:

> A man sees, grasps, or understands something no one else has seen...; ...if he wishes to share his discovery with other people.... Some other people must see, grasp, or understand what he has.... Some other people must accept that he has truly discovered a pattern.... A proof is constructed to get other people to accept the thing proved...

The hypothesisors have also influenced movements in mathematics education. Ormell's work as director of the Schools Council Sixth Form Mathematics Project reflects this philosophical stance most strongly. In the decade of the project's existence, related movements have arisen, such as the Mathematical Modelling Journal and Mathematics Applicable Group, with similar philosophic sympathies. There can be no doubt that the disputes about the nature of mathematics, that came to the fore, a hundred years ago, with the work of Frege, have and still do breathe life into differing movements in mathematics education. Given the clashes at all levels of one influence or another, particularly in published school mathematics materials, it is reasonable to expect teachers of mathematics to have knowledge and understanding of these philosophies that have helped to seed divisions.

In the second part of the thesis, it has been argued that four perspectives to mathematics teaching are identifiable. These perspectives are the Aesthetic, Game, Science and Technological orientations. Many great mathematicians have stressed the aesthetic-orientation. Zeeman and Hardy are just two that have been mentioned in this thesis. Many of the mathematics series of the nineteen-sixties stressed the games-orientation of mathematics teaching. S.M.P. has become both the most famous and the most popular. Contrasted with this orientation has been the unified methodology of Nuffield courses, that have emphasised the discovery method. This has been identified as paradigmatic of the Natural Sciences, but found in mathematics as well as physics, chemistry and other 'more strictly' science courses. Through the Schools Council mathematics courses has come the plea for the concrete and the practical. In many ways, these courses can be seen as turning back to the strongly instrumental views of courses that arose with the foundation of Secondary Modern education, as education for non-examination pupils. Daily Life Mathematics is just one such series that believed, 'the practical approach and visual methods of teaching are the most effective means of arousing' children's interests (Preface, p. vii).

These perspectives may not be mapped one to one mathematics series or mathematics teaching department, but there can be little doubt that the differing perspectives exist, and have voluble support. At least in part, the perspectives have been seen to develop from the philosophical frameworks identified in the first part of the thesis. Certainly, the increasing strength of the view of the nature of mathematics as the science of formal systems helped to provide stimulus to the commitment in the nineteen-sixties and early seventies for a 'narrow proof-like approach' in the presentation of mathematics in secondary schools. It was heralded as a closed and structured game, rather than the haphazard bag of recipes that had seemed to come across through the effect of the preceding influence, i.e. passing pure mathematics examinations. Much of the strength of the new approach seemed to lie in its endless expansion, rather as Descartes saw it when he wrote, 'making each truth that I discovered a rule for helping me to find others' (Discourse on the Method, p. 93, Haldane and Ross, 1967). The emphasis of formalisation was also an emphasis upon 'internal applicability', and contrasts with the balance of emphasis to be found here. In the technological orientation, outlined in part 2 Chapter 10, it was made clear that equal, and logically prior, importance is to be given to 'external applicability'. Yet, given that there are other views that have significant influence, it would seem foolish not to suggest that mathematics teachers should have knowledge and understanding of these perspectives too.

In the second part of the thesis, it has been made clear, as each perspective was introduced, how philosophical presuppositions about the nature of mathematics may limit the range of consistent stances that the mathematics teacher can take in regard to that perspective. Thus, any philosophical standpoint that relates to the nature of mathematics to 'proof in public use', puts clear limitations on the extent to which the games perspective can be developed.[1] The general point is that teachers ought to be expected to appreciate the interconnections among perspectives and philosophical movements. This is to reinforce the argument that both perspectives and philosophical movements ought to be known to mathematics teachers if they wish to claim an understanding of the full potential of mathematics in education.

It is not only methods of mathematics teaching and questions about the nature of mathematics that have been scrutinised in this thesis, but as has been discussed earlier in this part of the thesis, the aims of mathematics education, too. They may owe their selection, at least in part, to the influence of preferred philosophical standpoints as to the nature of mathematics, say. It would be unexpected, to say the least, to find that those who love mathematics for its lack of interpretations place external applicability as the central aim of mathematics education. It would be just as peculiar to find Ormell acclaiming that the greatest justification of mathematics in education lay in the creative forces that mathematics can help to unleash.

The expectation is that, given proper understanding of the methodologies and of the various views of the nature of mathematics, each

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1. The Hypothesisors come across as claiming that mathematics ends as well as begins in the real world, and this contradicts the essential notion of a game as non-serious and self-contained.

mathematics teacher can draw together through his own aims a coherent view of mathematics in his pupils' education.[1] Not all pupils can know what the teacher is required to know in order to make his pedagogic decisions, but the teacher ought to be prepared for 'whys' about <u>how</u> he teaches, and <u>what</u> he teaches.

Although the general tenor of this thesis has been to seek compromise, a preference has been indicated for education that is shown to be at least partly extrinsically valuable. The belief underlying this view of education is that education ought to provide the route to a more interactive[2] society, as well as being valued for its intrinsic worthwhileness. This approach has a logical consequence for all subjects, including mathematics. A subject can strengthen its place on the curriculum if it can be shown to be of both extrinsic and intrinsic value. Of all the perspectives found in part 2 of this thesis, it is the Technological orientation which most readily meets this possibility. In this one perspective, both forms of justification are necessarily present. The teacher is required to seek both internal and external applicability, in order to focus on the achievement of mathematical understanding in the <u>strong</u> sense.[3] Furthermore, this perspective nestles most kindly into one philosophical movement, that of the Hypothesisors,

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1. As Claude Birtwistle put it, nearly twenty years ago, 'Successful teaching of any subject depends on the methods of teaching employed, the background of the teacher and the content of the course.' (<u>Mathematics Teaching</u>, No. 17, 1961, p. 69).

2. 'Interactive' is used here to indicate the same sense as 'interactive application' has in the discussion on pp. 193-97 above, that people try to understand for themselves the point of view of others with whom they 'interact'; just as in mathematics it has been suggested that the mathematician should try to understand the area of knowledge to which the mathematics is applied and to be willing to modify both the mathematics and the subject to which it is applied.

3. On p. 247 above, the <u>strong</u> form of mathematical understanding was identified as involving at every point, the consideration of the external use of the given proposition, in addition to understanding connections within mathematics that relate to claims made about this proposition and its derivability.

without being totally incompatible with the others. Thus, a coherent mathematics education, founded on these pillars of Technological-orientation and mathematics as the science of possibilities, has much to commend it, but there is no question that in teacher training the duty lies in the provision of knowledge and understanding of all philosophies and perspectives as they relate to mathematics and education.

The conclusion reached is that the teacher is thereby able to use all approaches, and to explain to pupils that there are various views of the nature of mathematics. However, the limits of achievement for any one pupil at any one time must be left as a constraint on the teaching, that ultimately can only be resolved by the teacher himself, with support finally from 'learning theory' rather than 'philosophy'. The value of the programme in this thesis, it is hoped, is that its implementation would prevent future generations of pupils taking mathematics in secondary schools and coming out believing that there is only one view of the subject and one way for it to be presented. Under this programme, 'formal derivability' is seen as just one feature of mathematics, and presentation as a game as just one method of presentation, among many.

As Douglas Quadling put it, 'despite the effort expended on curriculum development over the past two decades, teachers of mathematics at secondary level still find themselves with fundamental unresolved questions of aim, of method and of balance. So far the experimentation which has gone on has fed on rich experience, but its achievement has been more to identify the questions than to provide the answers.' ('Issues at Secondary Level' in Mathematical Education, p. 180). It is to be hoped that this thesis has presented some answers and has not been satisfied with just presenting further questions.

APPENDIX

This appendix attempts to identify those theories of truth that have a clear influence on the philosophical movements discussed in part 1 of the thesis. It provides both background and an indication of how this author interprets these diverse theories, particularly as they arise in part 1, Chapter 5.

Theories of Truth. Aristotle's theory of truth is generally recognised as both the first theory and the forerunner of the Correspondence Theory. This theory has usually[1] been supported by realists,[2] while the other major theory of the first half of this century, the Coherence Theory[3] has had the support of anti-realists. Since the 1930s a new theory has taken the centre of the stage, Tarski's Semantic theory of truth. Even so, two further theories are described in this section because of their influence on the Philosophy of Mathematics. These are the Pragmatic theory which may be seen as somewhat of a compromise between the Correspondence and Coherence theories, having been initiated by the realist, Peirce, and modified by his anti-realist followers, James and Dewey, and the Redundancy Theory. The latter theory attempts to eliminate 'truth', and was founded by a philosopher of mathematics, Ramsey and at one point found favour with Wittgenstein.

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1. It is thus somewhat of a paradox that Austin provided a modified form of the theory.

2. Dummett provides a conception of 'realism' free from ontological commitment to mathematical objects in 'Realism' in Dummett, p. 146.

3. The Coherence Theory is a holistic theory that requires coherence among a complete set of beliefs. Taken in isolation it has found difficulty in collecting supporters but seen as a feature of a Pragmatic theory it is far more tenable. As no mathematical theory is to be discussed that holds to such an holistic theory of truth, any discussion of it will arise through the Pragmatic Theory (For a recent explication see, N. Rescher's The Coherence Theory of Truth, 1973 O.U.P.).

1) The Correspondence Theory of Truth

In its most cogent form, this theory argues that if a proposition is true, then there is a direct relationship between the meaning of the proposition and the facts being picked out by the proposition. Similarly, if the proposition is false then such a relationship will exist between the negative proposition and the facts. Thus, if 'the cat is on the mat' is true, then the cat is on the mat, and if 'the cat is on the mat' is false, then wherever the cat is, it is not on the mat. The traditional criticism of the theory as presented by Russell[1] and Wittgenstein[2] has been that the nature of the correspondence is unclear. If there is a claim for direct isomorphism between components of the proposition and components in the world then the theory would be rapidly refuted, for there may well be another language[3] in which the same proposition is expressed in far fewer or more words. Given this possibility there will be redundant components on some occasions and shortages on others, according to the language used. The theory only retains force if it is linked to linguistic conventions. In this form, to assert that 'the cat is on the mat' is to claim that this is a conventionally acceptable way of identifying a given set of facts[4] in the world. Hence, one can identify two forms of the Correspondence Theory. Cooper calls these, 'the

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1. 'On Propositions' in Logic and Knowledge, pp. 314-20, and 'On the Nature of Truth' in Philosophical Essays, pp. 149-59.

2. Tractatus, 4.01-4.1212. The criticism that follows, based on Cooper's argument Philosophy and the Nature of Language, does not apply to Wittgenstein, as Cooper admits, p. 96.

3. Cooper provides the fictional word 'Catamat' as a translation of 'the cat is on the mat' to highlight the problem for these earlier theories on 'components'. See Philosophy and the Nature of Language, p. 182.

4. In Austin's version of the Correspondence Theory, 'the facts' become synonymous with statements made in accordance with descriptive conventions. It would seem that this does not rule out the alternative use of 'facts' as referring to the world, rather than as statements about the world. Moreover Austin seemed to modify his position closer to the one suggested here, after an attack from Strawson. Compare his article 'Truth' pp. 121-4 with 'Unfair to facts', pp. 154-62, both in Philosophical Papers.

old-style theory'[1] and 'the new-style theory'.

The old-style theory claims a correspondence between a true sentence and facts in the world, according to a given ordering identifiable through the sentence. Thus 'the cat is on the mat' is distinguishable from 'the mat is on the cat', because of an ordering present in the sentence being identifiable, in some way in the facts.

The new-style theory presents 'correspondence' as dependent upon conventions that indicate how the meaning of the sentence, asserted as true, relates to things in the world. Cooper encapsulates the position as follows,

> "The cat is on the mat" is true if and only if the demonstrative[2] conventions governing the use of the referring expressions (e.g. "the mat"), and the descriptive conventions governing the use of the predicate (e.g. the cat must be on the mat), are both obeyed. To say that a sentence is true is to say just this - that the conventions in question are being obeyed. This is the sense in which "is true" serves to assert a correspondence of a conventional sort between sentences and the world.
> (Cooper, op. cit., p. 183. Footnote is not in the text).

Cooper admits that this is roughly the theory that Austin held. It provides criteria for some true propositions, though not a definition of truth, but it does at least show one how to be sure that a given assertion is true. It does not provide the meaning of 'p is true'. Furthermore, such a set of criteria would seem irrelevant in the case of analytic propositions, for in such cases nothing in the world needs to be demonstrated or described. Thus, this form of the correspondence theory may be an improvement on the earlier style, but it does not resolve all the problems about truth, and no doubt one would be extremely sceptical of any theory that claimed to do so.

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1. Cooper, op. cit., pp. 181-6.
2. Both 'the demonstrative conventions', by which correlation is achieved with general situations in the world (e.g. 'cat...mat'), and 'the descriptive conventions', by which correlation is achieved with particular connections among things as they occur in the world (e.g. 'the cat sat on the mat'), hook the theory on to the world.

2) The Pragmatic Theory of Truth

While Pragmatism is traditionally identified with Peirce, James and Dewey, the concentration here will be particularly on Peirce, as it is in the body of the thesis. However, the main features of Dewey's theory will be explained, because there would seem to be echoes of the theory in the views of contemporary philosophers of mathematics like Dummett,[1] who are dissatisfied with traditional 'realist' standpoints.

Peirce, James and Dewey agree on the strong link between the meaning of a concept, and the use to which it is put within the language. Truth is somewhat incidental to their overall theories, but they all focus on its connection with successful inquiry. As Peirce says, 'The opinion which is fated to be ultimately agreed to by all who investigate, is what we mean by truth...'[2] (Essays, p. 54). It is important to understand that Peirce is not saying that what is true is determined by what people generally accept at a particular time, nor that truth is just a matter of belief. While what one believes one believes to be true, belief is not a sufficient criterion of truth. In fact Peirce believed that any belief is fallible, except for those of logic and mathematics.

However, within ordinary language it makes sense to equate 'truth' with 'a state of belief unassailable by doubt' (C.P. 5.416) for truth itself can never be identified for certain (See p. 60 of the thesis for further discussion of this point). Now this state of belief is to be reached by 'a method...by which our beliefs may be determined by nothing human, but by some external permanency - by something upon which our thinking has no effect...the method must be such that the ultimate

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1. Dummett attempts to provide a philosophy of mathematics that values Intuitionistic logic without excluding necessarily a realist standpoint. See, 'The Philosophical Basis of Intuitionistic Logic' in Dummett, pp. 215ff.

2. In Peirce one finds 'inquiry' linked to the worthwhileness of a search for truth and in James and Dewey, it is explicitly of moral value. Definitely the Pragmatists indicate views now common in Philosophy of Education.

conclusion of every man shall be the same. Such is the method of science.' (Essays, pp. 24-5). This indicates the realism that James and Dewey were to reject while holding to the concern with inquiry. In Peirce's realism one finds similarities to a Correspondence Theory, for here too, there is a matching with 'Real things'. However, anyone's personal matching of a proposition to reality is not what makes the asserted proposition 'unassailable by doubt', but rather its being tested by others using the same method. In this way the proposition is either approved or adjusted.

Thus, truth is dependent firstly upon deriving propositions by the scientific method. These propositions indicate 'how things really and truly are' (Essays, p. 25), and secondly they are identified as true because as a matter of fact they achieve the support of an 'overwhelming consensus of opinion' (C.P. 6.610). The result is a rather thinly supported but nevertheless consistent theory of truth. This may be justified because Peirce was more concerned to lay down principles by which one could work, than to have a water-tight definition. While the Correspondence Theory may provide a better definition, it does not provide a working hypothesis and such passivity was an irritation to Peirce. As Ayer says of their position, Peirce and James conduct their enquiry'into the nature of truth from the standpoint of the individual thinker who is actually concerned with forming his beliefs. For someone in this position the cash-value of the question "What is truth?" is "How can I decide what propositions to accept?"' (The Origins of Pragmatism, p. 199). Tinged with Ayer's positivism, this does not indicate sufficiently the concern for achievement among the pragmatists. This is made even clearer in Dewey's position.

Dewey accepts Peirce's approach but can see no reason for calling beliefs 'true', if their status cannot be established conclusively. He prefers to talk of 'warranted assertibility'. The 'warranted assertion' is achieved at the end of an inquiry which involves the turning of an 'indeterminate situation', disequilibrium,[1] into a stable state, the solving of a problem. This is a procedure comparable to, but not identical with, Peirce's 'method of science'. This Dewey-type inquiry involves identifying the problem, establishing an hypothesis, theorising about and testing the hypothesis until the resolution of the problem is achieved. The proposition that states the conclusion to the inquiry is a 'warranted assertion', and the method of inquiry is the criterion by which the assertion is established. Thus the assertion, 'the cat sat on the mat' taken in isolation is meaningful[2] in only a weak sense. Unless one asserts that as the result of a problem like that faced by a distressed cat-owner in search of his cat, the proposition could not possibly be a warranted assertion. Links may be seen between Peirce's views and the Correspondence Theory on one side, and Dewey's views and anti-realist positions on the other.

Taking this last point further, one can try to identify more clearly the realist/anti-realist distinction that characterises Peirce's divergence from James and Dewey. In the Postscript (1972) to his article on Truth (1959), Dummett provides this clear identification of the distinction. He says,

> The fundamental difference between the anti-realist and the realist lies in this: that,...the anti-realist interprets "capable of being known" to mean "capable of being known <u>by us</u>", whereas the realist interprets it to mean "capable of being known by some hypothetical being whose intellectual capacities and powers of observation may exceed our own." ...The anti-realist holds...that the only meaning we can confer on our sentences must relate to those means of determining their truth-values which we actually possess. Hence, unless we have a means which would in principle decide the truth-value of a given statement, we do not have for it a notion of truth and falsity which would entitle us to say that it must be either true or false. (Dummett, p. 24).

1. This is indicative of Dewey's general use of biological and Darwinian analogies throughout his works.
2. In Pragmatist theories of meaning including Dewey's, a word gains meaning in use and so an isolated sentence is in a Pragmatic sense, 'meaningless'.

This quotation indicates clearly the realist/anti-realist gap and it will recur as the forms of mathematical truth become clearer. As the distinction becomes clearer, so will the problems of 'compromise' as attempted by Wittgenstein and Lakatos.

Dummett's own position in 1959 is clearly anti-realist, with clear echoes of Dewey when Dummett uses the term 'warranted'. He says that one is warranted in making an assertion only if one has the method by which 'we could in a finite time bring ourselves into a position in which we were justified either in asserting or in denying' the particular proposition (ibid., p. 16). Thus one has a pragmatic notion of truth linked to successful inquiry, but with two distinct underlying twists. Going one way, the realist way, one believes there are 'eternal answers', and going the other way, the anti-realist route, there are no answers except our answers.

3) The Redundancy Theory of Truth

This theory is usually attributed to F. P. Ramsey although both Dummett (ibid., p. 4) and Haack (Philosophy of Logics, p. 127) suggest it is derived from Frege but not supported by him. Ramsey argues that 'It is true that the cat is on the mat' has the same sense as 'the cat is on the mat', and so whenever a proposition is asserted no more is being said than the proposition alone. The conclusion drawn is that the phrase 'is true' can be eliminated. A similar elimination would occur for 'It is false that the cat is on the mat' by asserting 'the cat is not on the mat'. Ramsey rejects[1] any idea that 'true' or 'false' (as against 'negation') can be predicated of a proposition, so their use in language can only be for stylistic reasons or emphasis. Certainly the employment of these words has led to paradoxes that would otherwise be eliminated, and one wonders whether, like Tarski, Ramsey was concerned

1. 'Facts and Propositions' in The Foundations of Mathematics, pp. 142-4.

to remove the Liar paradox, one version of which is 'The sentence on line 2 page 280 of this Appendix' is false. Ramsey was certainly interested in the paradoxes and categorises them into semantical ones, like the Liar one and purely 'logical' or mathematical ones, like Russell's paradox, concerning 'The sets of all sets that are not members of themselves'. (If S is a member of S then there is a set in which S _is_ a member of itself and which _is not_ a member of the set of _all_ sets that are not members of themselves). It may be that Ramsey thought that the way in which semantical paradoxes were due 'to faulty ideas concerning thought and language' (_ibid_., p. 21) was because it had not been appreciated that 'true' and 'false' could be eliminated.[1]

The main difficulty of Ramsey's theory is that one _does_ find 'true' and 'false' contributing to the meaning, and not just the emphasis of propositions. As was noted on p. 23 above, there is a difference between the semantic principle of bivalence, 'every statement is either true or false' and the logical law 'p v -p'. This distinction evaporates for Ramsey, as the semantic principle can have no other interpretation than 'p v -p'. In other words, if 'true' and 'false' add nothing to the meaning of sentences, then they are either 'p' or '-p' according to their form and that is that. Yet there are propositions like, 'Truth is stranger than fiction'[2] and 'Everything he says is true'. It is much harder to make 'truth' or 'is true' redundant in these examples than in 'it is true that the cat is on the mat'. 'True' does not here occur predicated of a proposition, so that it is not clear how according to the redundancy theory, it is to be eliminated. The theory is still valuable because it causes one to ask the questions, 'Do we need "true" and "false" in

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1. Naturally, not all paradoxes involve 'true' and 'false'.
2. This example is used by Platts, _Ways of Meaning_, p. 11 to indicate one of the other uses of 'truth' not covered by Ramsey's theory, but here it exemplifies two points. 'Truth' in use and the problems of verifying certain propositions, are indicated.

ordinary language, and do we need "true" and "false" in formal languages?' The argument so far has been that the answer to the first question is indubitably positive, and the work of Tarski and implicitly Goedel's, has been to facilitate answers to questions about its necessity or otherwise in formal languages.

4) The Semantic Theory of Truth

In his work on truth[1] Tarski saw himself reiterating Aristotle's original definition and consolidating it, rather than firing a revolution.[2] Tarski provides criteria which must be satisfied for any definition of truth to be adequate, and he also exemplifies this by a definition of truth for a particular formal language. It is important to remember that he provides criteria for a definition and not criteria for truth.

The first criterion of material adequacy is that for every sentence S of the language L (for which a theory of truth is to be constructed), there is a provable theorem in the theory which takes the form:

(T) S is true iff p (iff is 'if and only if')

(where 'S' is a structural descriptive name of the sentence p).

Davidson[3] gave this form to Tarski's 'Convention T,[4] and the example made famous by Tarski, 'Snow is white' is true iff snow is white.

The second criterion is that there is a finite set of logical principles to be employed to deduce instances of T, and deduced from a finite set of basic axioms. The possibility of producing such theories of truth is

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1. His theory is found in two articles: 'The concept of truth in formalised languages' in Logic, Semantics and Metamathematics and 'The semantic conception of truth' in Readings in Philosophical Analysis (ed. Feigl H. and Sellars W.).

2. Tarski refers explicitly to Aristotle's definition as a source, 'Concept of Truth in Formalised Languages', p. 155, footnote 2. In the work of Davidson the result has been like a revolution in the seventies, comparable to that sparked off by the works of Wittgenstein in the sixties.

3. See for example, Davidson's reference to it as 'Tarski's convention T' in a footnote in 'On Saying That' in Words and Objections, p. 173.

4. In 'Concept of Truth in Formalised Languages', pp. 187-8, Tarski introduces the phrase, 'Convention T'.

severely limited by the requirement that every sentence is to have its proven (T) sentence, for it is quite clear that most languages, including formal ones, have an infinite number of sentences. This production of potentially infinite sentences is dealt with in formal languages, by building up chains under a recursive principle. That is to see new sentences as conjunctions of already given sentences, which correspond to conjunctions of (T) sentences. Thus

'If S is true iff p and T is true iff q then S and T is true iff p and q'

or, 'If "Snow is white" is true iff snow is white and "Grass is green" iff grass is green then "Snow is white and grass is green" iff snow is white and grass is green'.

This explains only the barest bones of Tarski's theory, but to do more would require technical elaborations that are inessential to the argument of the thesis as a whole. However three further points do seem worth making:

a) Tarski believes that no theory of truth for an ordinary language can be formed to meet his criteria, because ordinary languages will always contain self-referential paradoxes like that on p. 280 above. These can be avoided in a formalised language, which has distinct Object-language and Meta-language.[1]

b) Dummett criticises Tarski's theory for only giving sense to truth in the form, '<u>is</u> true', and not in any other sense, i.e. tense or mood. This is not just a shortcoming for ordinary language, but also for formal languages like modal logics. As Dummett says, 'its role in our language does not reveal why such inflections of tense or even mood should be forbidden'. (Dummett, p. 233).

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1. Taking a simple reflexive proposition like 'this is a lie', 'this is a lie' is true iff this is a lie. If the Right-Hand-side and the Left-Hand-side belong to different languages, then there cannot be a paradox, for the reflexiveness is retained within each side.

c) Popper and Platts[1] both argue convincingly that Tarski's theory is a reformulation of the Correspondence Theory, for links are made in the Tarski theory between what carries the truth: a sentence, and what tells one that the sentence is true: entities in 'the world'.

Some Further Comments. In this thesis, topics have been considered that cover volumes of words in philosophy, and no less can be said of 'truth', than has been said of 'realism', 'science', 'aims', and so on. It is hoped that this appendix has given finger-nail sketches of theories of truth whose features are recognisable in the main body of the thesis. It should not be thought that the absence of significant discussions of the views of Davidson, Kripke, Wiggins, et al. indicates either a lack of interest or ignorance of their views. It does indicate however, the belief that the main argument of this thesis has been clearly presented without the need for additional considerations of 'truth' and 'meaning' as these writers view them. The author is confident that whatever else these recent views have identified, they have not provided unanimity on these matters. It is sufficient for the argument of this thesis, that the reader accepts that there are generally consistent but different[2] ways of considering such key concepts, and that in an elementary form, these differences have been indicated here.

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1. See Popper, Unended Quest, pp. 141-4 and Platts, Ways of Meaning, pp. 34-7. Haack in Philosophy of Logics, pp. 112-14 employs a similar line to Platts, but concludes more cautiously, 'Tarski's definition of satisfaction, if not of truth, bears some analogy to correspondence theories'.

2. Theories presented separately here may have been brought under one roof by philosophers in the last fifteen years or so. Certainly, Strawson has indicated that the Correspondence Theory (new-style) and Ramsey's Redundancy Theory may have sufficient overlap that they can be regarded as variants of one school. Strawson presents such a possibility in 'A Problem about Truth' and 'Truth: A Reconsideration of Austin's views' (Logico-Linguistic Papers, pp. 214-33, and pp.234-49). Brought together one can view this theory as one alternative to Davidson's view that 'truth' in ordinary language is undefinable, except in conforming to Tarski's Convention T, from which criteria for the use of 'truth' are identified. The developments in Davidson's position are identifiable in such articles as, 'Truth as Meaning'

[Contd. overleaf

Fn. 2, p. 283 contd.

(*Synthese*, xvii, 1967, pp. 304-23), when compared with 'Thought and Talk' (*Mind and Language*, ed. S. D. Guttenplan, Clarendon Press, 1975, pp. 7-23) or 'Reply to Foster' (*Truth and Meaning*, ed. G. Evans and J. McDowell, O.U.P., 1976, pp. 33-41). Davidson has come to be seen as representing the view that theories of meaning can only be devised from the successful construction of a theory of truth, and not vice versa. Other alternatives to this position include,

a) Dummett's view which has been shown in this thesis to be closely tied to an Intuitionist standpoint, in which a theory of meaning presumes the identification of a theory, not of truth, but of 'constructibility', or 'verification' or 'falsification'.

b) Quine also provides an alternative that has significant sympathy with a position taken centrally in the thesis, that of the Pragmatists, particularly those like Peirce who have realist inclinations. Quine also fails to find a guarantee of unambiguous communication in truth conditions. He accepts the 'indeterminacy of translation' as support for his belief that any security for communication is to be found in the possibility of empirically equivalent experiences that people have, rather than in what they say. In the discussion of the analytic/synthetic distinction in Chapter 5 of this thesis, this point is implicit in the position identified for Quine. As with Dummett's position, the successful use of 'truth' is found in what people do, and this form of realism was discussed on pp. 276-79 above.

While Dummett stands alone in allowing the possibility of anti-realism, he indicates a sympathy with those influenced by Austin, Strawson and more strongly recently, by Grice, who believe that Davidson's (and incidentally Quine too) position may provide an adequate, if restricted, theory of truth, but in using such a theory to frame a theory of meaning, the position stands against that of ordinary usage. The elimination of intensional presuppositions, as occur if one claims that it makes sense to talk of '*the* meaning of "Eric Blaire is alive"', as if one would talk of abstract entities, is considered by Grice and others as tantamount to the elimination of meaning itself. Certainly, Davidson only has room for 'meaning in L', rather than 'meaning' *per se*. Apart from the truth conditions by which one can test the truth of 'Eric Blaire is alive', one is left, according to this rebuttal of Davidson, with only the alternative of 'this is what I/she/he/etc. means by...', and not, 'this is what is meant by...'. There is the soft echo of Plato and Frege, as they have been met in the main body of the thesis, in this alternative to Davidson.

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Mathematics, Science of Possibility*—A Critical Review

E. BLAIRE†

Holland Park School, London, England

Summary

This paper supports the view that search for the applicative standpoint in mathematical education has yet to be sufficiently exploited. Mathematical propositions are considered to have both an 'internal' and 'external' role and thus a system of 'potential models' is evolved. Despite the stress on the applicative nature of the subject it is argued in conclusion that the position in the body of the paper is compatible with the synthetic apriority of mathematics.

What is Mathematics About?

'Mathematics, Science of Possibility'[1] is the phrase Ormell uses to entitle his answer to the question, 'what mathematics is about?' I support Ormell's answer but feel that some of his arguments may gain from expansion and critical appraisal. His is a genuine attempt to show on philosophical grounds that the recent approach to mathematics, the isolationist policy, is not being true to what mathematics is about. Mathematics is not just an aesthetically pleasing manipulation of symbols but a purposeful activity. To the mathematical educationist, particularly to those interested in the less able, the recent dragging of mathematics higher into the clouds is most frustrating. The less able child is at a totally concrete level of thought, and for him, learning mathematics can be validated only as a purposeful activity. Ormell's own interests are towards the A level child and so if both extremes of mathematics cry out for purposeful mathematics, then there are good grounds for looking for a justification of mathematics as such. If it could be shown that mathematics is mainly valuable as an artistic game, then surely many of the views of education inherent in our society would be turned on their heads. Mathematics has status today not for intrinsic value as an activity but because of the purposeful and applicative nature of the activities within it.

Unfortunately just because one feels that it would be good for education if such-and-such were the case, it does not make it the case. This article is intended to be as impartial as possible, and to fault as well as support Ormell's arguments and conclusions.

It may be that one does not have to make a choice between mathematics as the production and manipulation of axiomatic systems and mathematics as 'the study of what is true of hypothetical states of things'.[2] Tucker[3] suggests that mathematical propositions have both an internal and an external role. As the former, they are rules, proofs, identities, etc. of a fact-free but meaningful nature, and as the latter, they have the potentiality for being models of possible pictures of reality. Tucker rejects the formalist

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† Assistant Master in Mathematics.

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position where he says that mathematics consists of meaningless axiomatic systems, but supports for his internal role, formalizers where the axiomatic systems are meaningful. Whether or not such rigid formalists ever existed is not to be debated here, but I take mathematical meaningfulness to infer the following:

$a+b = c$ has a meaning given in terms of the relations and the relating of variables by the relations. It is not necessary for the proposition to have substitution incidents in order that it should become meaningful in a formalist sense, that is true or false.

What is to be considered here is whether Ormell has provided a notion of mathematics which puts these roles in a truer perspective. In one form or another the vast majority of people would accept that mathematical propositions do have this 'internal role'. However, a purely internal debate cannot give a justification for mathematics in a wider context. The 'internal role' is no answer to the question 'what are mathematical propositions used for?' as asked by the outsider. He wants to know the purpose of mathematics and he expects quite rightly an ordinary language justification. As science is generally justified by the value of its application, so may be mathematics, but the arguments cannot be identical. Mathematics, the 'hand-maiden' of science is one more step away from reality—what is constructed.

Any but those concepts taken as fundamental are open to operational definition in science. The concepts of mathematics are at most open to a purposive definition for 'externally' they do not 'do' anything. The 'objects' of the mathematical world are themselves passive in function. The 'objects' of the scientific world have themselves an active function. Pi has a purpose, but electrons do things. As soon as one tries to externally justify mathematics, one turns to science for comparison. People like Peirce, Popper and Lakatos have tried to use one theory to explain both disciplines. Both Peircean Fallibilism and Popperian Falsificationism stress the importance of counter-example. Tucker has noted that not all mathematics as an activity is on the level of constructive proofs, but much is on the level of constructive arguments. It is on this less rigorous level that the work of Lakatos[4] (following Popper) seems to hold water. While the mathematical world searches for proof, counter-examples are valuable. Peirce saw that this process cannot be taken as identical to that in science, for mathematics consists of fact-free hypotheses. Mathematics refers only to '*possible* states of affairs'. One might argue that mathematics at its lowest level is like science at its highest, for generally science is validated by observation and by correct prediction of a specific nature.

Model

It would seem at this point that one might take up Ormell's position. To him, mathematics is about problem-solving and it does this by a simulating activity. The hypotheses of Peirce become the models of Ormell. Ormell uses the word 'model' not in its usual mathematical sense but in its scientific sense. His 'model' is not an 'internal-role' player but an 'external-role' player. It is not the kind of model that shows the consistency and independence of the axioms of a deductive system, but the kind of model used by Wittgenstein[5] in his *Remarks on the Foundations of Mathematics* where $2+2 = 4$ is a model of a possible situation. (The in-between model are strokes made on paper or fingers counted to discover how many dolls a girl has.) This model helps to solve a problem in 'the external world'. It is not just a sterile paradigm. The line between mathematics and science may seem to disappear with this approach but what may remain is the distinction between a hypothetical and an experimental study. A little girl is experimenting to find out how many dolls she has, laying them out in front of her and counting them by one method or another. Her answer is an empirical statement that may be false. This is not doing mathematics because mathematics is not an activity of manipulating objects as they are given. Anyone who recognizes any type of distinction between mathematics and science accepts this. A formalist might say that the mathematician deals with problems of the 'outside world' by idealization. What seems important for Ormell's thesis is to make clear the distinction between a simulating activity and an idealizing activity. The general position seems to have been to see the application of pure mathematics to experience as an activity to be regulated from within pure mathematics.

Let us take a hideously simple example:

'How many apples are there when I add an apple to a box of apples with one apple already in it?'

The logicist knows the statement '$1+1 = 2$' is an identity statement or an analytically true proposition. The problem must be reduced to a statement that is able to be substituted by a logical symbolic phrase deducible from the axioms (plus those necessary additional ones) of the system. Suppose we get '1 apple add 1 apple = 2 apples'. This may now be said to characterize units and couples thus having the same logical status as the original arithmetic proposition. The formalist takes this a stage further and ends up with a totally symbolic statement like

$$((x_0 \epsilon 1) \;\&\; (y_0 \epsilon 1)) \equiv ((x_0 \cup y_0) \epsilon 2)$$

which has the same logical status as

$$(x)(y)((x \epsilon 1) \;\&\; (y \epsilon 1)) \equiv ((x \cup y) \epsilon 2)$$

These approaches and the intuitionist are all inward looking and in no way explain the application of mathematics to experience, to the lay-man. They are not explanations at all but further layers of 'private language'. Körner[6] and others have tried to break out of this network by saying that the mathematical statement is an idealization of the problem. Every property of the apples is ignored except for that of being of 'unit' value. The 'adding' relationship between the apples is retained, but no questions are asked of how or why the adding comes about. Thus we come to a mathematical solution '$1+1 = 2$' which when de-idealized gives me an answer to my problem, 'two apples'. Körner represents this as

$$e_1 \wedge (e_1 = b_1) \rightarrow (b_1 \rightarrow b_2) \rightarrow (b_2 = e_2) \wedge e_2$$

What we have are empirical situations put through an idealizing machine to shrink them down to their 'exact concepts' and a reverse process at the end of the operation. To Körner[7] the statement '1 apple and 1 apple makes 2 apples' treated in this way is a problem of applied mathematics which is itself inexact and not distinguishable from theoretical physics say. What is important is that the process of idealization from inexact to exact concepts is thought to be necessarily the way applied mathematics functions and also it should be noted that logicists and formalists think their descriptions are similarly unique.

At least Körner has tried to get outside to deal with the problem, but he has not recognized the hypothetical status in practice of applied mathematics problems. If the question is 'how many apples does 1 apple and 1 apple make?' it may be mathematical but if it is, 'here is 1 apple and here is 1 apple, how many apples are there?' then the question is not for the mathematician but for the man in the street or the scientist. The mathematician deals with the hypothetical, not the actual. Only the actual needs to be idealized, not the hypothetical. Thus it is that Ormell's simulating activity of mathematics in application to experience is far more digestible than an idealizing theory. If one deals with possible situations then one is taking the elements of reality as they stand without a need to pass them through idealization processes for they are to be modelled in the mathematics not 'processed' into it. One looks for applicable mathematical models for the problem situation and sees which provides the best practical solution. What one is not claiming is that there is a discipline of applied mathematics which is governed by necessary rules so as to process all problems of reality with mathematical content. It is seeing applicable mathematics in this light that leads to the rigid traditional views of applied mathematics. There is nothing there but the pure mathematics, the real life problem and a way of looking at it and tackling it. What does not exist in any case is applied mathematics. One has pure mathematics which is the source of models for simulating real-life problems if interpreted in the right way. This method of interpretation is not an extension of pure mathematics, governed by rules and a logico-symbolic system but rather, an extension of imagination—another external rather than internal source of the aesthetic in mathematics.

Interpreted Mathematics

Let us try again to clarify how mathematics is to be described. It is not to be taken as uninterpreted symbolism for if this is all that it is then educationally, unless one is to be a formalist, it has almost no value in the external world. However, if it is to be 'interpreted mathematics' then one must enquire for what purpose is it interpreted. The answer here is to say that the formalist has produced an infinite number of 'kits' of uninterpreted symbolism from which mathematical models of situations may be made up. These are all 'potential models' each of which may have an infinite number of possible situations in which it may be interpreted. To see this as the connection between mathematics and reality seems fine, but Ormell seems to have a hint of the idea that within mathematics itself this 'modelling' is on-going. That is to say that mathematics is a layered cake where the uninterpreted symbolism ('uninterpreted' for the formalist and 'interpreted' for the formalizer) may in one sense be a model of the uninterpreted symbolism at a higher level of generality. When one talks in this way one may be twisting from the concept of a 'scientific model' to that of a 'mathematical model' without showing either that the two concepts are essentially one or admitting the change.

I shall take it that what Ormell means is that any level of mathematics may be used as a model for a solution of a problem depending on the level of generality of the problem. Thus any part of mathematics may be interpreted and so it is mathematics *in toto* which is the 'science of possibility'. Ormell discusses the language of possibility and it seems to me that this language is most applicable to 'action' rather than 'objects'. The 'what' of possibility may just be an emphasis of English where the only available general pronoun is not as neutral as those of other languages. The language and the reality of imagination on the other hand necessarily have an objective primacy. To imagine building a wall depends, if we are to do it in mental pictures, upon starting with objects that then do things. It seems to me that it is the possibility of building a wall which is analogous to the possibility of mathematics and not the possibility of a wall. Ormell is trying to see mathematics as a next level of abstraction from disciplined imagination but this may have no more than a twisted validity. By disciplined imagination I may go through the complete process of building a wall but the plans of the architect, quantity surveyor or builder may be more abstract. What distinguishes the imagined processes from others including mathematics is the lack of possibility of verification or falsification. This is not an attack on Ormell for he does discuss how mathematics differs from scale drawings, etc. It may be noted that Ormell says that what makes mathematics fundamental here is that 'there is nothing in addition to the record' and Wittgenstein makes a similar remark, in the *Remarks on the Foundations of Mathematics*: 'In mathematics process and result are equivalent'. Wittgenstein goes on to make another relevant point in distinguishing what mathematics allows us to do and science does not.

'I can calculate in the medium of imagination, but not experiment.' Lakatos talks of 'thought experiments' and sees them only differing from their scientific counterparts in their levels of abstraction, while Wittgenstein thought there was a logical gap. I side with Wittgenstein in what follows and believes that he would have had great sympathy for Ormell's views. Wittgenstein equates mathematical solutions and models at one point and talks of mathematical propositions as frameworks of possible descriptions.

'Mathematics forms a network of norms.'

It seems to me that Wittgenstein believed mathematics is important anthropologically because it has application and if it had no application it would have far less status.

'The mathematician is an inventor, not a discoverer.'

Only the unknown or ill-considered artist acclaims that art has its true value in the creating and not in its communication. This is the failing that Ormell is clearly highlighting. Even earlier in Peirce[8,9] we find similar lines of thought and perhaps Aristotle's always so practical spirit lies back in the depth of this program.

To return to the surface I feel that Ormell[10,11] might have validly laid greater stress on 'interpreted mathematics' as a purposeful activity. That is to say the manipulation, the simulating, the connecting is to be stressed rather than the things related. It seems to me that much of the sterility lies in the ghost of logicism which is a set theory/class program as against a 'program of relations'. Ormell talks of 'reification' which has a thing-orientated connotation. This may be better seen as 'the possibility of states of affairs'. Let us see mathematics as a theory of possible relations. In a problem about a wall and a ladder, set so that I can get over the wall, if I am in prison the wall, the ladder and the setting are not themselves just possibilities. What is found in 'disciplined imagination' are the possible ways these things may be related and similarly my mathematical model provides me with ways of relating things. The mathematics is not so much 'a study of what is true of hypothetical states of things' as Peirce put it, but rather 'the hypothetical relationships among things'. Thus, one may be able to make a direct attack on 'idealizing theories' of applied mathematics by stressing the place of relations. Ormell rightly states that in mathematics the record and the manipulation are one, but too often it has been forgotten that the power of mathematics lies in the possible manipulations rather than in the expressions so produced. The manipulations cannot be idealizations of activities in the real world but are simulating activities comparable to a computer program simulating population growth. Reproduction cannot occur in a computer. Mathematics is itself the investigation of possible relationships between fact-free symbols.

Thus, one has two notions of mathematics as the science of possibility. In its non-applied state it is providing, as Ormell said, 'kits' and then related to a particular problem one has a set of possibilities some of which may satisfactorily simulate the needed relationships of the problem. It is in the latter interpretation that the life-mathematics dichotomy is solved, or at least given a public hearing.

Interpretations

Before drawing conclusions I should like to discuss this interpretation of mathematics with regard firstly to education and secondly to science. Ormell notes the distinction between statements of applicable mathematics and those of a mixed number/verbal nature. A statement of the first kind would be open to genuine simulation by mathematics while the second type is not 'intended' for such. This is partly to go back over what has been previously stressed, that in the past philosophers have used statements of the second type to exemplify applied mathematics. They have taken it that these were just simplified versions of 'the meaty problems of reality'. These may not however be mathematical statements in the first sense at all. Take a statement such as:

'six defenders and five attackers make up a football team'.

This may be simulated as

$$6a+5b = c \quad \text{or} \quad 6p+5p = 11p$$

but my sporting comment only becomes a question of mathematics if the context is appropriate. That is to repudiate someone who agrees that there are eleven players but thinks there are seven defenders and five attackers. The point is that Ormell says that numbers may be just symbols independent of any mathematical framework and I want to take this further and say that one may make a statement which for you is not an attempt to make any correct numerical comment. I may not be able to add up to eleven and yet still know:

'six defenders and five attackers make up a football team'.

Relating to this are several educational points that are often ignored. Firstly a child is counting correctly when he says: 'one, two, three, ...', not because he is making valid manipulations but merely because he repeats a socially accepted verbal procedure. Similarly, a child may learn to count up to a hundred and that ninety-nine is followed by one hundred but not that 2,135 follows 2,134 or that 2,136 means two thousand, one hundred and thirty-six. It is not just a matter of set theory for the child but also things like recursive definitions, a theory of relations and the learning of a public means of precise communication.

Perhaps more important than these examples is how the purity of mathematics as stressed by logicists and formalists, etc. can lead to practically speaking dangerous or at least deplorable attitudes for adult life. The war-cry of method and not correct calculation is not necessarily right. One may be able to rely on a computer for calculations but one should recognize that errors of one part in a million may spell doom. In the classroom one may tell pupils at A level that the constants of indefinite integrals are not to be calculated, but in problems of aircraft friction it is the 'C's that matter and not those easily obtained beautiful pieces of integration.

Thus mathematics may be aesthetically pleasing but the educationalist must not ignore its applicative value. We have seen that mathematics and science may often go hand-in-hand. However, Ormell suggests that a possible distinction lies in the fact that what one might call mathematical experience and the results derived from it are therefore 'α-repeatable', while that of science would only be 'β-repeatable'. Körner talks of pure mathematics having 'infinite-repeatability' in a similar way. Ormell does not want to draw such a dramatic line between mathematics and science. It may be just that they considered themselves scientists that the symbolic models of Maxwell, Sommerfeld, Bohr and Einstein were seen as parts of theoretical physics. No one can deny that it was interpretation within physics that gave experimental value to the models. Bohr's circles and Sommerfeld's Keplerian ellipses are closer to the imagination and thus mathematics than experience. Hermann Weyl said that physics attempts to convert natural laws into *a priori* constructed mathematical functions. Perhaps it is more correct to see functions provided which are capable of interpretation within physics.

Mathematics is still not verified by experience and as such remains logically distinct from science. However, the same mathematics as an interpreted model may be shown to be a correct model by experimentation. That is to say that the correctness of our choice of model from all the possible ones is validated by experimentation, but not the model. Whatever roles mathematics may have, there remains only one discipline of knowledge, but there may be several fields of interpretation.

Conclusions

In conclusion I want to lay emphasis on two features of Ormell's 'Mathematics, Science of Possibility', the one seems of particular educational importance and the other of particular philosophical importance.

Firstly, if one accepts that mathematics is a simulating activity for problem-solving in life-mathematics contexts, then this is the most practically relevant way a child can see what mathematics is about. If Peters is right and education is initiation into worthwhile activities, then there seems to have been a danger recently of seeing mathematics as 'initiation into an applicationless metaphysics or mind-trainer' which not even the Pythagorians would have seen as adequate justification for giving mathematics a central place in the scheme of things. It is one thing to value art for its own sake but another to deny society the value of mathematics as a 'science'.

Secondly, Ormell seems to feel that the underlying support given to the philosophical thesis that mathematics is synthetic *a priori* is no longer of importance, whether right or wrong. That one knows $3+5=8$ only by experience and that the proving and the recording are one, does not seem to me to show that mathematical knowledge is not logically different in status from scientific. Ormell may well have shown that the Kantian position is most unsatisfactory, but surely mathematics as 'fact-free manipulation' at a level of abstraction above 'disciplined imagination' is still distinct by more than just degree from the discipline of science. The choosing of 'kits' may be empirical but surely not the kits themselves. What may be being brought to light is the strange status of much of the work of theoretical sciences. These too may be 'kits' whose construction is also comparable to 'disciplined imagination' but they are not constructed with the intention that they should hold a truth independent of experience. To someone who relies on the notion of applied mathematics as an idealization of science, a distinct structure from pure mathematics, then the theoretical sciences may be interpreted similarly. However, by Ormell's conception of mathematics, one is not forced into that position and there are no propositions of applied mathematics with a status lying between the *a priori* and the *a posteriori*, for all mathematical propositions can keep their *a priori* status. Only the interpretations may be open to empirical validation. Similarly, there are no inexact concepts of applied mathematics to worry about and a similar saving may be found for the concepts of theoretical sciences. As possible models they can be treated as possessing concepts no less exact than those of mathematics, but they are distinct from those of mathematics because no one intends them to have both an internal and an external role. There is no purely symbolic uninterpreted physical science but only the interpreted kind. A proposition of mathematics can only be validated necessarily internally while a proposition of physics can at most be internally falsified—its validation is necessarily external and as such empirical. In this sense mathematics retains its logical priority and independence of the theoretical sciences.

What Ormell may have left for others to show is that while mathematics is a 'science of possibility', branches at least of other sciences may also be given this title, although I would add, not the same logical status. It is not the intention of this paper to decry the value of traditional philosophies of mathematics for their internal analyses of mathematics, but only to doubt their value as external explanations and thereby their value for justifying mathematics as one among other worthwhile activities within education. Ormell himself seems to go one step further and to say that the Peircean-based philosophy of mathematics is a philosophy of mathematics while the approaches of logicists, formalists and intuitionists are not, because they are nothing but internal analyses, metamathematics and not philosophy.

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Creativity and Mathematical Thinking

E. Blaire

In this article Eric Blaire discusses Prof. Zeeman's view that mathematics is at once 'inhuman' and 'creative art'. Prof. Zeeman's article in Mathematics in Schools *was, however, reprinted from the* Psychiatric Quarterly, *1966; it may not fully represent Prof. Zeeman's current views.*

This paper may be best seen as a reply by a school-teacher to an article by Zeeman.[1] In that article mathematics is claimed to be unique among the sciences, as it alone is both creative and independent of humanity and the universe. It is neither my wish nor belief to deny that mathematics is open to aesthetic appreciation, but I fear that Zeeman's expression of his position is open to dangerous misinterpretations at levels below that of the University.

Zeeman is probably right to draw distinctions between Mathematics and the Arts, but perhaps the intended lines should have been drawn more clearly. He says, "It is as if every art student of today could copy and improve upon the canvasses of the great masters". But why stick to such a narrow conception of the Arts? Logically literary masterpieces could be repeated, and some abstract paintings are open to identical reproduction. The analogy seems particularly troublesome if we consider prints or moulds which the student can reproduce and perhaps undeniably improve upon in, say, clarity by means of technical advances. Certainly Zeeman wishes to make a logical, rather than an empirical, point. Perhaps the point is that necessarily mathematical 'masterpieces' have a duality. That is to say, they are not what they are only for themselves, but also because they are new tools for the continuation of the game. As such they may be used and improved by anyone working in the same field. New uses may be found for them within the subject. It is this 'instrumentalism' which draws a clear line between Mathematics and the Arts.

Elliott's distinction between the traditional and new concepts of creativity[2] may provide us with a similar line. If mathematicians are not 'makers' but 'discoverers' then on Elliott's view one would not apply the traditional concept of 'creative' to what they do. However the new concept finds its paradigms in practical activities where exceptional imaginativeness or ingenuity is denoted as 'creative' in this new sense. It may be that the Arts mainly use one concept and the Sciences the other, and Mathematics which at various times has been thought to belong to either, neither or both camps, may have had both concepts applied to it. Zeeman certainly seems to intend to use the traditional concept while putting mathematics into the sciences where the traditional concept is inapplicable. Thus if we see mathematical creativity only in the new sense this kind of analogy problem will not arise.

Thus far mathematics and the sciences have not necessarily been differentiated. Zeeman uses the concept of 'simplification' to do this. Unfortunately Zeeman does not prove that it is logically impossible for this process to go on in the sciences except through mathematics. It may be logically possible for some sciences to simplify themselves without any substantial use of mathematics. It may, of course, have been Zeeman's intention to define 'simplification' as a mathematical activity, but he does not explicitly say so. It seems to me that many scientific theories may have had more complex forms in the past than they have now. Examples of this are found in the theory of evolution, changes in zoological and botanical taxonomies and in geology. There one can find simplification without any primary requirement of involvement of mathematics.

The power to simplify and explain, when applied to the natural and social sciences, is not necessarily the main basis for Society's support of mathematics. It may be argued that mathematics is more readily appreciated for its power to penetrate obscure ranges of possibilities, *e.g.* in the sciences. Zeeman can only turn to his own cultural group to confirm his commitment to the belief that the main justification for doing mathematics ought to be its "creativity and elegance". This does not prove that applicability is only a secondary justification.

Zeeman is demanding ultimately an emotional tie to mathematics. We have seen above that it is not possible convincingly to argue that mathematics is "the most original and most creative of all the sciences", but another point arises here: Zeeman has not shown that mathematics does not fall within 'the technologies', rather than the sciences.

If the applicative standpoint is central to mathematics then perhaps the basis for its* growth may lie in its strength as a technology, rather than as a science. (This is not an argument, but a thought for consideration.) This could be another way of highlighting the problem of the use of the word 'creative'; for in some sense 'pottery' may advance through being a 'technology' connected to metallurgy, a 'science'. Similarly mathematics may be open to such division; and in one area, the technological one, the traditional concept of 'creative' *is* appropriate, whereas in the scientific area only the new concept is appropriate. Perhaps one can go further and say that the activity of "creative thinking" as found in mathematics (which we may not agree can logically be divided into two parts) is distinguishable in its denotation from that found in any other area. That is, at least as a matter of fact, it is the only area where thinking is concerned with models of possible 'worlds' rather than actual. If this were shown to be the case, then Zeeman's use of 'most' would be, to say the least, inappropriate.

Most people, of course, are not mathematicians, but there *is* an undeniable need for more people to have some

**I.e.* the growth of mathematics

competence in, and understanding of, mathematics. If emphasis on the aesthetic value of mathematics is intended to bring this about by means of the educational process, then Zeeman is making some deep demands on Society. On Zeeman's position initiation into mathematics is understanding (if not 'acquiring') mathematical 'taste'; but this is at a level of mathematical competence that only a few attain. If others are to be so initiated then these few would seem to have ahead of them an inhuman task.

Certain questions now arise about the possible relationship between 'the aesthetic' and mathematics. The qualities of "elegance, intrinsic beauty, profundity, generality, simplicity, depth, subtlety and economy" which make up mathematical 'taste' in Zeeman's opinion may be found in aesthetic appreciation generally. However Zeeman does not say whether or not these qualities are to possess particular features which may characterise them as applicable to mathematics alone. All these qualities may equally apply to music, say. Would Zeeman be happy to see music replace mathematics on the School time-table, if it were shown that at an earlier age a student may acquire greater appreciation of these qualities in music than in mathematics? If 'the aesthetic' is to be a justification for doing mathematics, then *either* it cannot be a major one, *or* the aesthetic qualities named above must in some respects be either unique or peculiarly highlighted in mathematics. The last point may, of course, be a statement of Zeeman's position, but he does not say so explicitly. Perhaps 'the aesthetic' is not to be mainly taken as a justification for teaching mathematics, but as an aim. That is to say, that at least the understanding of these qualities as exemplified in mathematics is one of the important ways a student can be shown to be initiated into mathematics.

If we were discussing music this might seem to be a fair position; for no one doubts its place among the Arts; but with regard to mathematics there seem to be various problems.

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Firstly, we have agreed that mathematics is not an Art. Secondly, Dienes for example, may accept the importance of these qualities in teaching mathematics as 'aims'. However, he may think that such qualities can be understood and exemplified by a child at a considerably lower level of mathematical development than that characterised by Zeeman. There may be a resulting confusion of different concepts by the primary and secondary teacher. Thirdly, at the secondary level in particular, one may question whether or not 'the aesthetic'can be taken as a major aim (or aim at all) in teaching mathematics. Surely, if we were to stress what may be highly subjective features of mathematical education, then the result could be a chaotic variation of material covered. The main aims of a secondary teacher must surely be to equip his students for their further needs in the area of mathematics, and also for those areas to which they may apply their developing mathematical knowledge. Certainly 'The Aesthetic' may be an extrinsic aim of the teacher, but the proper consideration of it as an intrinsic aim would require greater clarification of the qualities laid down by Zeeman.

Zeeman is not unaware of the applicative value of mathematics, and talks of valuable models to be produced in relation to brain cells. What is worrying is the tendency of modern text books to stress 'imagination, ingenuity and originality', which are at least some of the features of 'creative thinking' in mathematics (using Elliott's new concept), without sufficient regard to applicability, which is, at least, equally important. If one has only the notion of 'ingenuity', say, as found in 'the great mathematicians' then one is aiming at a level totally inappropriate to the average secondary school child. In the child's eyes this is likely to 'come across' as a logical, hard-headed approach to the subject which will alienate the vast majority of eleven-year-olds, possibly for life. On the other hand, an approach which has a considerable amount of applicativeness may provide the foundation necessary for producing students interested in mathematics.

A teacher must recognise that his students are capable of a level of thinking distinct in complexity, and probably in quality, from that of say, Zeeman. To use an analogy, it is not only the material that changes as the student advances, but also his tools. In this case the tools are, to a great extent, 'ways of thinking and approaching problems'. The teacher may use 'material' which does not fall within the compass of pure mathematics. With guidance the child may develop 'his tools' through the manipulation of blocks whose physical properties are essential for the solution of the problem. Mathematics can be done before a child reaches a level of 'abstract thinking'. There is a danger that if one concentrates on seeing mathematics in its 'abstract, inhuman' form one may blind oneself to the need for its initiation in a concrete form at various levels of conceptual development.

As the title of this paper implies, I am not attacking Zeeman's views as such, but I am urging caution: because some of them may be taken as dogmas into the classroom, where they have, and I would say, must, lead to trouble. One finds attempts to teach eleven-year-olds to understand infinite cardinals, because the teacher wants to take his students along a logical sequence 'to the end'! It is forgotten that the child has yet to gain the equipment for such a level of conceptualisation. If all he possesses are concrete conceptions of mathematics, then he will be totally incapable of grasping Zeeman's abstract aesthetic qualities which the conceptually developed person may be capable of.

What may be independent of 'levels' is the kind of approach demanded by mathematics. That is to say, given a problem, one 'kicks it around' mentally and then one relies on trial and error (plus experience) to discover possible solutions.

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When one finds an answer one cleans it up and if it still does hold, one has one more model to call upon in the future. (And if not, one tries again.) Zeeman is highlighting what one finds in a 'real mathematician', and these qualities may be taken possibly by teachers as an ultimate goal which can be met only *if*, at the lower levels, the ground is prepared and the seeds sown. The danger then may be to treat the goal as a set of objectives capable of achievement in some way or other, at every level.

Unfortunately 'thinking' is a difficult concept to define, and in schools 'thinking mathematically' requires at least two different interpretations. In each there is some opening for 'creativity', but it is to be found most readily where the 'isolationist mathematician' will not look. There is, firstly, what Dienes calls, "constructive thinking". But this is not half as important as 'analytic thinking'. Under this heading can fall the whole field of 'tackling problems'. A teacher has taken his students a considerable way when they can recognise which problems are suitable for mathematical treatment - perhaps this is what needs to be stressed most with regard to mathematical initiation. This might not be called a part of mathematics teaching at all from the point of view of a 'Zeeman-type philosophy' for it is an indubitably 'human' activity.

Given a problem as mathematical, then the student must be able to pick out those factors which are relevant (and sufficient) for formulating an adequate model (which may not solve the problem, even then). As Miss Edith Biggs has always made clear, most of the 'creativity' goes on prior to computation and/or manipulation of symbols. (This is most probably the view of professional mathematicians like Zeeman, also.) While some 'structural learning' (following a strictly logical sequence) may be necessary in order to operate in this area, there is no evidence, of either a logical or a psychological nature, to suggest that mathematics as a whole, demands such an approach. It is just an impression given by people putting mathematics once formulated, on a level with propositional calculus. The truths there set down may be "independent of individual preferences" but this alone is not evidence that as an activity it possesses a unique objectivity, particularly for the student whose evaluation of mathematics may differ both from that of a teacher and/or a mathematician.

Mathematics has been considered by too many people as an almost mystical activity where the uninitiated can tread only with the most calamitous results. Zeeman is putting mathematics on just such another pedestal; and any pedestal gives many teachers and students the impression that it is out of their reach. "These views lead to self-fulfillment!" Only the 'genius' attempts to study on these terms at any depth because everyone believes that it is the subject for 'geniuses' alone.

Zeeman links the poetic with orderliness, reliability, and predictability, but the link is ultimately subjective. It may be important to initiate children into mathematics - as Sawyer thinks - so that one cannot say of an answer: "If they had thought what it meant, they would have seen that it was ridiculous". The stress here is for *understanding*, and must not be taken as a cry for creative appreciation. The latter is only possible logically once there is sufficient understanding. 'Creativity' as a concept certainly possesses aesthetic overtones, but to recognise the importance of "simplification" is one thing and to make beauty a criterion of mathematical truth or validity is another. Zeeman as a mathematician may be able to afford to dull the distinctions of ordinary language, but the teacher needs to be clear about the difference between the traditional and the new concepts of 'creativity'.

Let us make one final expression of belief in the humanity of mathematics. The history of mathematics is full of instances of human influences. They may be logically only indirect but they yet remain essential. What guarantee is there that it would develop in the same way in inhuman or machine minds? The dominance of the denary system is a daily reminder of physiological pressure. That the Greeks developed geometry and not another branch of mathematics seems almost certainly to have been the result of an inefficient numerical recording system. Gattegno argues: "Mathematics thus reflects, like any other human activity, the personal, the individual qualities which make its future as completely open as it must have been in the past". Mathematics reflects the societies of its origin. Recent developments in foundation theory all seem to indicate just how *human* rather than inhuman this subject is. People now ask, just how much of the rigour is an illusion: they complain that systems have been "forced" into logical frameworks by human inventiveness. The limits of mathematics lie there - these are now the fundamental questions of today.

In concluding this article I return to 1944, the point from which rapid changes in School Mathematics were to come. In the report of that year on the teaching of mathematics one finds the desire that mathematics should provide "vivid and practical parts of the pupil's experience" and also that "every stage in the working of every problem should be set out in the form of a logical and grammatical statement; and taken together, these should fit consecutively into a coherent, meaningful and grammatical whole", This is again an attempt to link creativity and objectivity. Here the 'logical' seems to be synonymous with 'common sense' or perhaps worse, 'what is right'. Certainly these are worrying beginnings which still seem to be echoed. No matter what one means by 'logical' it is dangerous to take as synonymous for mathematics phrases like 'creative thinking' and 'logical coherence'. I believe such mistakes are made by teachers partly because of the lack of clarity on the part of people like Zeeman, who fail to emphasise that the 'juggling, groping, hitting and missing' that precede discovery are not even necessarily logical activities (at all) and thus cannot be identical with the cold symbols that ultimately make up the new proof. The proof is not a written record of creativity; it is just a sign that creative thinking has gone on. It does not tell one what creative thinking is like.

Zeeman has described his experience; the danger is to try to bring about the same level of experience independent of a student's conceptual development; to initiate creative thinking as if it were logical thinking, and *vice versa*. No one should confuse the need for stimulating ingenuity, as capable of leading to mathematical accomplishment, with the teaching of logical rigour. Zeeman clearly expects the one to follow the other. I have tried to clarify the causes of this confusion and to urge the need for care when we are introducing mathematics at school level. E.B.

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